

TAMING THE POLYNOMIAL EXPLOSION: A NEW APPROACH TO ALGEBRAIC CIRCUIT VERIFICATION

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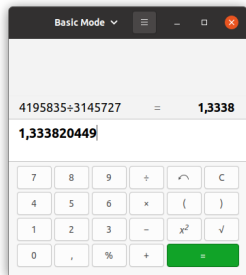
Joint work with Jérémy Berthomieu, Sorbonne Université, CNRS, Paris, France

Seminar Talk
Stanford University, CA, US

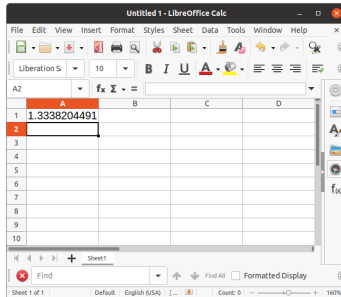
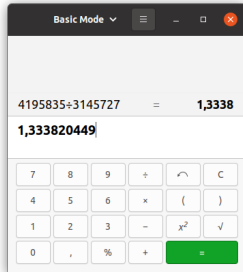
May 12, 2025

$$4\,195\,835 \div 3\,145\,727 = ?$$

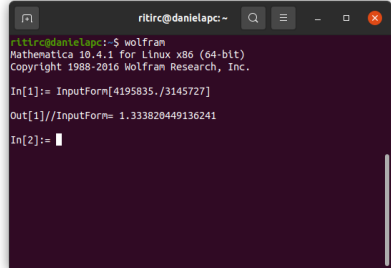
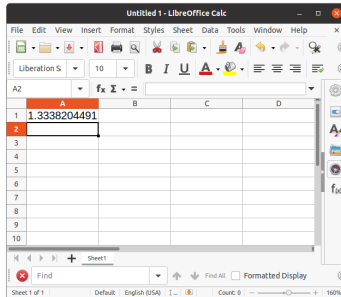
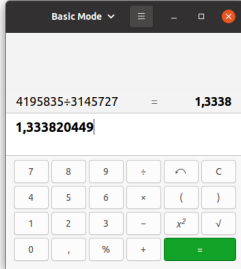
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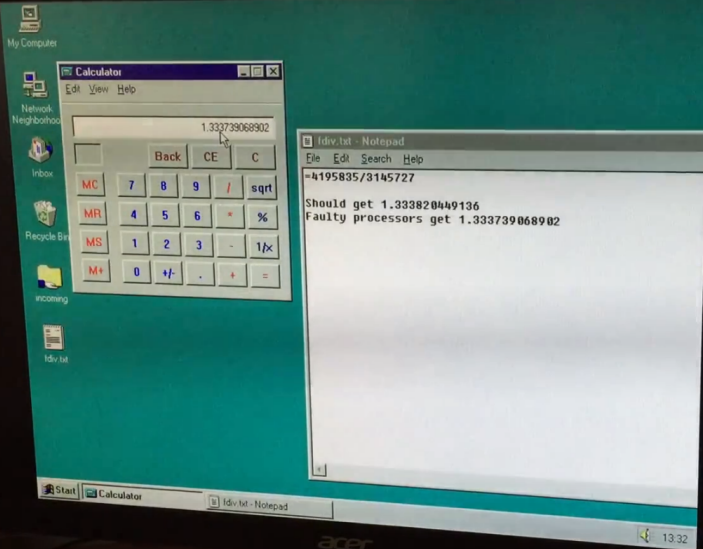


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Intel Pentium FDIV-Bug 1994



Quelle: <http://neology.com.au/portfolios/a80502-90-sx923/>

- Affected floating point unit (FPU) in early Intel processors.
- Processor might return incorrect result for division.
- Cost in 1994: 500 million dollars.

Even more than 30 years later verification of arithmetic circuits is considered to be hard. Correctness proofs are not fully automated yet.

Challenge: Integer multipliers

Integer Multiplication

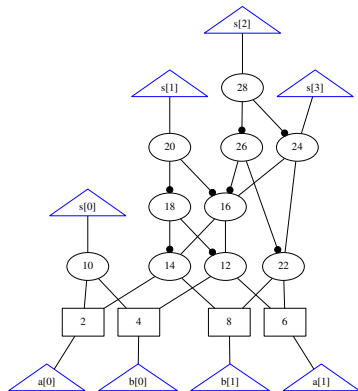
$$\begin{array}{r} 11 \cdot 11 \\ \hline 11 \\ 110 \\ \hline 1001 \end{array}$$

$$3 \cdot 3 = 9$$

Integer Multiplication

$$\begin{array}{r}
 11 \cdot 11 \\
 \hline
 11 \\
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 \hline
 1001
 \end{array}$$

$$3 \cdot 3 = 9$$



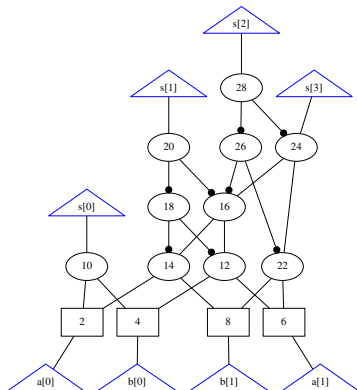
And-Inverter Graph

Multiplier Circuits

Given: Gate-level multiplier for fixed bit-width.

Question: For all possible $a_i, b_i \in \mathbb{B}$:

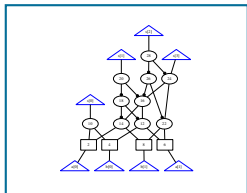
$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$



And-Inverter Graph

Formal Verification

System



Specification

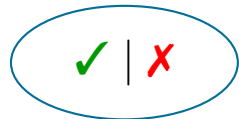
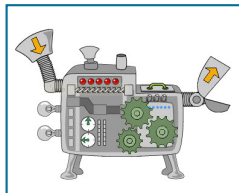
For all $a_i, b_i \in \mathbb{B}$:

$$(2a_1 + a_0) * (2b_1 + b_0) =$$
$$8s_3 + 4s_2 + 2s_1 + s_0?$$

Mathematical Model

$$B = \{$$
$$x - a_0 * b_0,$$
$$y - a_1 * b_1,$$
$$s_0 - x * y,$$
$$\dots$$
$$\}$$

Automated Decision Process



Formal Verification Techniques

Decision Diagrams

- First technique to detect Pentium bug

[ChenBryant DAC'95]

- Requires knowledge of the layout

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Theorem Proving

- Used in industry, e.g., ACL2

[TemelSlobodovaHunt CAV'20]

- Automated on the RTL level

[Temel TACAS'24]

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Satisfiability Checking (SAT)

- SAT 2016: Exponential run-time of solvers [Biere SATComp'16]

- SAT 2024: Equivalence checking of structural similar circuits

[BiereFallerFazekasFleuryFroleyksPollitt SATComp'24]

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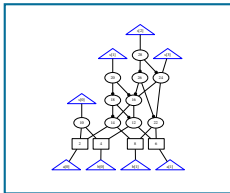
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Algebraic Approach

- Seminal work: [LvKallaEnescu TCAD'13, CiesielskiYuBrownLiuRossi DAC'15]
[SayedGroßeKühneSoekenDrechsler DATE'16]
- Polynomial encoding
- Works for non-trivial multiplier designs

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$$B = \{ \\ x - a_0 * b_0, \\ y - a_1 * b_1, \\ s_0 - x * y, \\ \dots \\ \}$$



Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \\ \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$



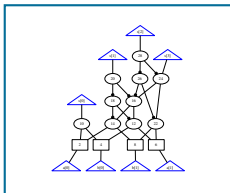
Implication

$= 0$ ✓

$\neq 0$ ✗

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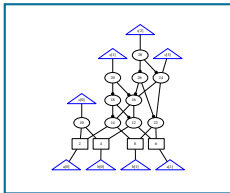


Ideal Membership

$$\begin{array}{l} = 0 \text{ } \checkmark \\ \neq 0 \text{ } \times \end{array}$$

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Multiplier Specification

Unsigned Integers:

$$0 = \sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right) \in \mathbb{Z}[X]$$

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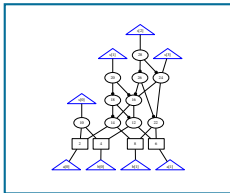
$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right) \in \mathbb{Z}[X]$$

Signed Integers:

$$-2^{2n-1} s_{2n-1} + \sum_{i=0}^{2n-2} 2^i s_i - \left(-2^{n-1} a_{n-1} + \sum_{i=0}^{n-2} 2^i a_i \right) \left(-2^{n-1} b_{n-1} + \sum_{i=0}^{n-2} 2^i b_i \right) \in \mathbb{Z}[X]$$

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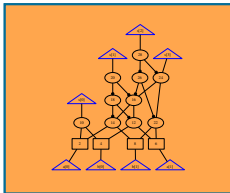


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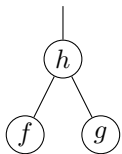
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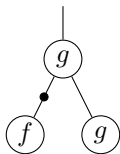
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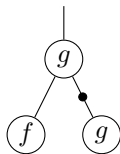
From AIGs to Polynomials



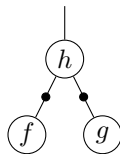
$$h = f \wedge g$$



$$h = \neg f \wedge g$$

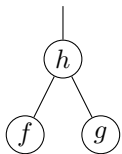


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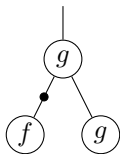
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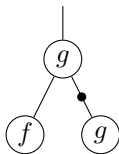
$$h = f \wedge g$$

f	g	h
0	0	0
0	1	0
1	0	0
1	1	1



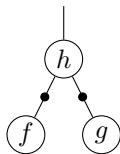
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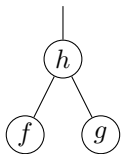
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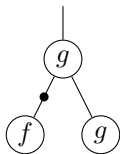
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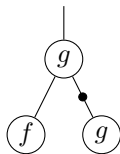
$$-h + fg$$



$$h = \neg f \wedge g$$

f	g	h
0	0	0
0	1	1
1	0	0
1	1	0

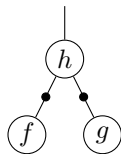
$$-h - fg + g$$



$$h = f \wedge \neg g$$

f	g	h
0	0	0
0	1	0
1	0	1
1	1	0

$$-h - fg + f$$



$$h = \neg f \wedge \neg g$$

f	g	h
0	0	1
0	1	0
1	0	0
1	1	0

$$-h + fg - f - g + 1$$

From AIGs to Polynomials

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

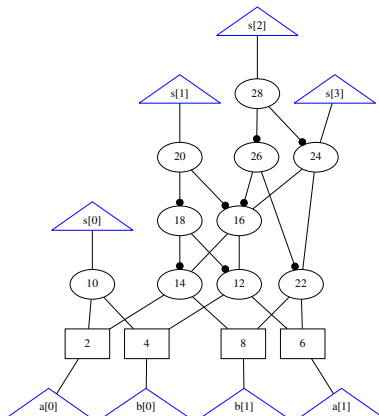
$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0b_1$$

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$$-l_{10} + a_0b_0$$



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Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

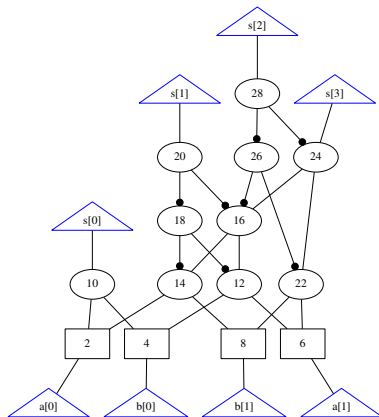
$$\begin{array}{ll}
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 -s_1 + l_{20} & -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\
 -s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
 -l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
 -l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\
 -l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0
 \end{array}$$

Boolean input constraints $B(C) \subseteq \mathbb{Z}[X]$.

$$\begin{array}{ll}
 a_1, a_0 \in \mathbb{B} & -a_1^2 + a_1, -a_0^2 + a_0, \\
 b_1, b_0 \in \mathbb{B} & -b_1^2 + b_1, -b_0^2 + b_0
 \end{array}$$

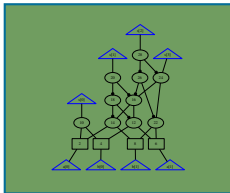
Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1 a_1 - 2b_1 a_0 - 2b_0 a_1 - b_0 a_0$$



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Ideal Membership

$= 0$ ✓

$\neq 0$ ✗

COMPUTER ALGEBRA

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IDEALS AND GRÖBNER BASES

Ideal

Ideal. A subset $I \subset R[X]$ is an ideal if it satisfies:

- $0 \in I$
- If $f, g \in I$, then $f + g \in I$.
- If $f \in I$ and $h \in R[X]$ then $hf \in I$.

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Ideal generated by a finite number of polynomials.

Let $f_1, \dots, f_s \in R[X]$. Then we set

$$\langle f_1, \dots, f_s \rangle = \{h_1 f_1 + \dots + h_s f_s \mid h_1, \dots, h_s \in R[X]\}.$$

$\langle f_1, \dots, f_s \rangle$ is an ideal and is called the ideal generated by $f_1, \dots, f_s$.

Hilbert Basis Theorem. Every ideal has a finite basis.

Ideal

The ideal $\langle f_1, \dots, f_s \rangle$ has a nice interpretation in terms of polynomial equations.

Given $f_1, \dots, f_s \in R[X]$, we get the system of equations

$$f_1 = 0,$$

$$\vdots$$

$$f_s = 0.$$

Let $h_1, \dots, h_s \in R[X]$. We can derive $h_1 f_1 = 0$, $h_2 f_2 = 0$, $h_1 f_1 + h_2 f_2 = 0$ etc.

Hence we obtain $h_1 f_1 + \dots + h_s f_s = 0$ as a consequence of our initial system.

Thus, we can think of $\langle f_1, \dots, f_s \rangle$ as consisting of all “polynomial consequences” of the equations $f_1 = f_2 = \dots = f_s = 0$.

Applications of Ideals

The Ideal Membership Problem.

Given $f \in R[X]$ and an ideal $I = \langle f_1, \dots, f_s \rangle \subset R[X]$, determine if $f \in I$.

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Given $f \in R[X]$ and an ideal $I = \langle f_1, \dots, f_s \rangle \subset R[X]$, determine if $f \in I$.

Reduce f by $f_1, \dots, f_s$?

Ideal Membership Problem

Let $I = \langle x - y, x - 2 \rangle \subset \mathbb{Q}[x, y]$.

Is the polynomial $f = xy + x - 3y \in I$?

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Spoiler: Yes, because

$$xy + x - 3y = (x - y) + y(x - 2)$$

Ideal Membership Problem

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Is the polynomial $f = xy + x - 3y \in I$?

$$xy + x - 3y \xrightarrow{x-y} y^2 - 2y$$

$$xy + x - 3y \xrightarrow{x-2} y - 2$$

Operation $\xrightarrow{p}$ is multivariate variant of polynomial division.

Special Bases

Given a set of polynomials F in $R[X]$.

- Transform F into another set $G \subset R[X]$ **with certain nice properties** such that $\langle F \rangle = \langle G \rangle$.

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- Transform F into another set $G \subset R[X]$ **with certain nice properties** such that $\langle F \rangle = \langle G \rangle$.
- A whole range of problems defined for an arbitrary set of polynomials F becomes algorithmically solvable using G .
- G is called a **Gröbner basis** [Buchberger'65].

Properties of Gröbner Bases

Lemma 1. Every ideal $I \subseteq R[X]$ has a Gröbner basis G w.r.t. a fixed monomial order.

Lemma 2. Given an arbitrary basis, we can compute G .

Lemma 3. Every $f \in R[X]$ has a **unique computable remainder** $r \in R[X]$ with respect to G such that no term in r is divisible by any of $\text{lt}(g_i)$.

Lemma 4. $f - r \in \langle G \rangle$.

Hence, $f \in \langle G \rangle$ iff the remainder of f with respect to G is zero.

Orderings on the Monomials

Univariate Polynomials - Sort by Degree.

$$\dots > x^{m+1} > x^m > \dots > x^2 > x > 1$$

Example: $7x^5 + 5x^4 - 2x^3 + x - 6$

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Linear Polynomials - Sort by variables.

$$x_1 > x_2 > \dots > x_n$$

Example: $8x + 3y - 3z$ for $x > y > z$

Orderings on the Monomials

Univariate Polynomials - Sort by Degree.

$$\dots > x^{m+1} > x^m > \dots > x^2 > x > 1$$

Example: $7x^5 + 5x^4 - 2x^3 + x - 6$

Linear Polynomials - Sort by variables.

$$x_1 > x_2 > \dots > x_n$$

Example: $8x + 3y - 3z$ for $x > y > z$

How to order non-linear multivariate polynomials in $R[X]$?

Orderings on the Monomials

Requirements:

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- Multiplication: For any monomials σ_1, σ_2, τ , we require that $\sigma_1 < \sigma_2 \implies \sigma_1 \tau < \sigma_2 \tau$.
- Well-Ordering: Every nonempty subset of monomials has a smallest element.

Orderings on the Monomials

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Lexicographic Order $\prec_{\text{lex}}$.

$\sigma_1 \prec_{\text{lex}} \sigma_2$ iff there exists an index i with $u_j = v_j$ for all $j < i$, and $u_i < v_i$.

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Degree Reverse Lexicographic Order $\prec_{\text{drl}}$.

$\sigma_1 \prec_{\text{drl}} \sigma_2$ iff either $|\sigma_1| < |\sigma_2|$ or $|\sigma_1| = |\sigma_2|$ and $\sigma_2 \prec_{\text{lex}} \sigma_1$.

Example: $xy^3 < xy^2z^3$

Leading Elements

Let f in $R[X]$ be ordered w.r.t to an ordering $<$ such that

$$f = a_1\tau_1 + a_2\tau_2 + \dots + a_m\tau_m.$$

Then we call

- $\text{lt}(f) = a_1\tau_1$ is the **leading term** of f .
- $\text{lm}(f) = \tau_1$ is the **leading monomial** of f .
- $\text{lc}(f) = a_1$ is the **leading coefficient** of f .
- $f - \text{lt}(f) = a_2\tau_2 + \dots + a_m\tau_m$ is the **tail** of f .

Computing a Gröbner Basis

Algorithm: Buchberger's Algorithm

Input : $F = \{f_1, \dots, f_s\}$, monomial ordering $<$

Output: Gröbner basis $G = \{g_1, \dots, g_t\}$ w.r.t. $<$, such that $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_t \rangle$

$G = F$;

$C = \{\{g_1, g_2\} \mid g_1, g_2 \in G, g_1 \neq g_2\}$;

while not all pairs $\{g_1, g_2\} \in C$ are marked **do**

 choose unmarked pair $\{g_1, g_2\}$;

 mark $\{g_1, g_2\}$;

$h = \text{normalform of } \text{spol}(g_1, g_2) \text{ w.r.t. } G$ $(\text{spol}(g_1, g_2) \xrightarrow{G} h)$;

if $h \neq 0$ **then**

$C = C \cup \{\{g, h\} \mid g \in G\}$;

$G = G \cup \{h\}$;

return G

Product Criterion. If $p, q \in k[x_1, \dots, x_n] \setminus \{0\}$ are such that the leading monomials are coprime, i.e., $\text{lcm}(\text{lm}(p), \text{lm}(q)) = \text{lm}(p) \text{lm}(q)$, then $\text{spol}(p, q)$ reduces to zero mod $\{p, q\}$.

Ideal Membership Problem

Let $I = \langle x - y, x - 2 \rangle \subset \mathbb{Q}[x, y]$.

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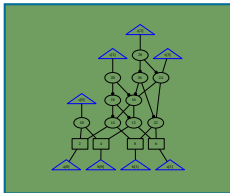
$$xy + x - 3y \xrightarrow{x-y} y^2 - 2y \xrightarrow{y-2} 0$$

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BACK TO CIRCUITS

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$$B = \left\{ \begin{array}{l} x - a_0 * b_0, \\ y - a_1 * b_1, \\ s_0 - x * y, \\ \dots \end{array} \right\}$$

Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$

Ideal Membership

$= 0$ ✓

$\neq 0$ ✗

Ideal Membership Problem

[RitircBiereKauers FMCAD'17]

■ Polynomial Encoding:

- Gate polynomials $G(C)$
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- Lexicographic term order: Output variable of a gate is greater than input variables.

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$G(C) \cup B(C)$ is a Gröbner basis for $J(C)$.

Proof idea: Application of Buchberger's Product criterion.

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Multiplier. A circuit C is called a multiplier if

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right) \in J(C).$$

Verification Algorithm

Reduce specification $\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right) \left(\sum_{i=0}^{n-1} 2^i b_i\right)$ by elements of $G(C) \cup B(C)$

until no further reduction is possible, then C is a multiplier iff remainder is zero.

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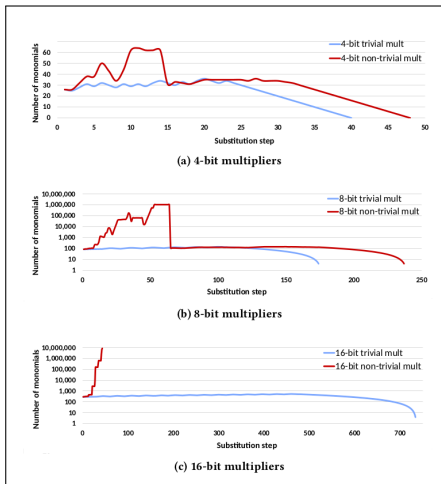
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Computational Problems

- The number of monomials in the intermediate results blows-up.
- 8-bit multiplier cannot be verified within 20 minutes.
- Reason: Degree of substitutions is not bounded!

Verification Algorithm

[MahzoonGroßeDrechsler ICCAD'18]



Strategies

1. Encoding

- Embedding different phases [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]

2. Preprocessing

- Variable Elimination [MahzoonGroßeDrechsler DAC'19, RitircBiereKauers DATE'18]

3. Reduction

- Incremental Algorithm [RitircBiereKauers FMCAD'17]
- Dynamic Reduction Order [MahzoonGroßeSchollDrechsler DATE'20, KonradScholl FMCAD'24]

4. Tricky: OR Gates in final stage adder

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All of these strategies rely on a **lexicographic term ordering**.

Change of Order - TACAS 2025

	$\prec_{\text{lex}}$	$\prec_{\text{drl}}$
GB Computation	✓ Easy	⚠ Hard
Spec Reduction	⚠ Hard	✓ Easy

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If the specification polynomial is linear,
a Gröbner basis with respect to a
degree reverse lexicographic term ordering
contains linear polynomials that suffice
to derive correctness of the circuit.

Theorem

Theorem

Let $p \in \mathbb{K}[X]$ with $\deg(p) = 1$, $I \subseteq \mathbb{K}[X]$ be an ideal. Let G be a Gröbner basis of I with respect to $\prec_{\text{drl}}$ and let $G_1 = \{g \in G \mid \deg(g) \leq 1\}$. We have $p \in I$ if and only if $p \rightarrow_{G_1} 0$. In particular, $p = \alpha_1 g_1 + \cdots + \alpha_m g_m$ with $g_i \in G_1$, $\alpha_i \in \mathbb{K}$.

Linear Gröbner Basis Reduction Algorithm

Algorithm: Linear Gröbner basis reduction

Input : Circuit C in AIG format, Specification polynomial S

Output: Determine whether C fulfills the specification

$G_{\text{init}} \leftarrow \text{Gate-Polynomials}(C) \cup \text{Boolean-Input-Polynomials}(C);$

$S_{\text{lin}}, G_{\text{ext}} \leftarrow \text{Linearize}(S);$

$G_{\text{drl}} \leftarrow \text{Compute-}\prec_{\text{drl}}\text{-Gröbner-Basis}(G_{\text{init}} \cup G_{\text{ext}})$

$G_1 \leftarrow \{g \mid g \in G_{\text{drl}} \wedge \deg(g) \leq 1\};$

while $\text{lm}(S_{\text{lin}}) \in \{\text{lm}(g) \mid g \in G_1\}$ **do**

$p_{\text{lin}} \leftarrow g \in G_1$ such that $\text{lm}(g) = \text{lm}(S_{\text{lin}});$

if $\nexists p_{\text{lin}}$ **then return** $\perp$;

$S_{\text{lin}} \leftarrow \text{Linear-Reduce}(S_{\text{lin}}, p_{\text{lin}});$

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$$\mathcal{S}_{\text{lin}}, G_{\text{ext}} \leftarrow \text{Linearize}(\mathcal{S})$$

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

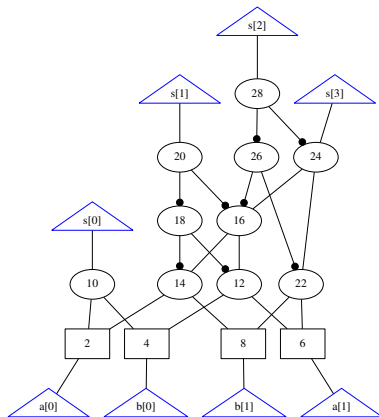
$$-l_{14} + a_0b_1$$

$$-l_{12} + a_1b_0$$

$$-l_{10} + a_0b_0$$

Specification $\mathcal{S} \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1a_1 - 2b_1a_0 - 2b_0a_1 - b_0a_0$$



$$\mathcal{S}_{\text{lin}}, G_{\text{ext}} \leftarrow \text{Linearize}(\mathcal{S})$$

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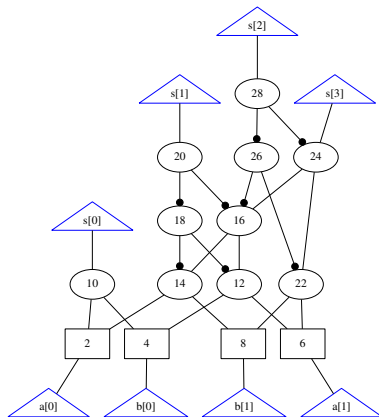
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Extension polynomials $G_{\text{ext}} \subseteq \mathbb{Z}[X]$.

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Linear Specification $\mathcal{S}_{\text{lin}} \in \mathbb{Z}[X]$.

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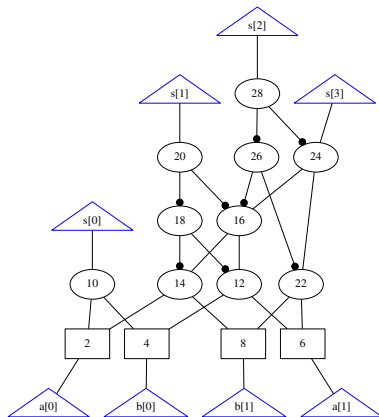
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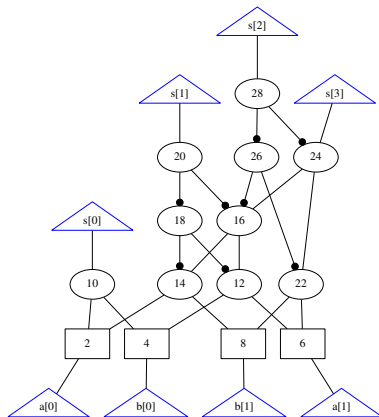
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$\text{Preprocessing}(G_{\text{ext}});$

while $\text{lm}(S_{\text{lin}}) \in \{\text{lm}(g) | g \in G\}$ **do**

$p \leftarrow g \in G$ such that $\text{lm}(g) = \text{lm}(S_{\text{lin}});$

$p_{\text{lin}} \leftarrow \text{Linearize-Single-Polynomial}(p, G);$ // On-the-fly

if $p_{\text{lin}} = 0$ **then return** $\perp;$

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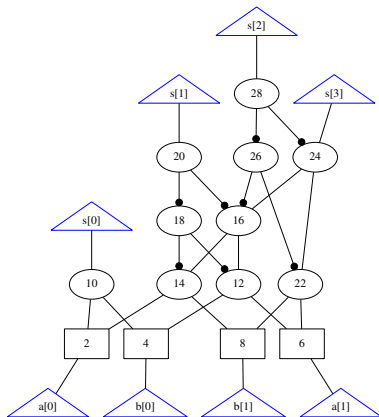
On-the-fly Linearization

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 -s_1 + l_{20} & -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\
 -s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
 -l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
 -l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\
 -l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0
 \end{array}$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$



On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0b_1$$

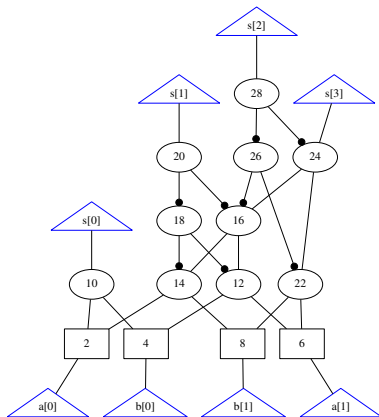
$$-l_{12} + a_1b_0$$

$$-l_{10} + a_0b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$



On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0b_1$$

$$-l_{12} + a_1b_0$$

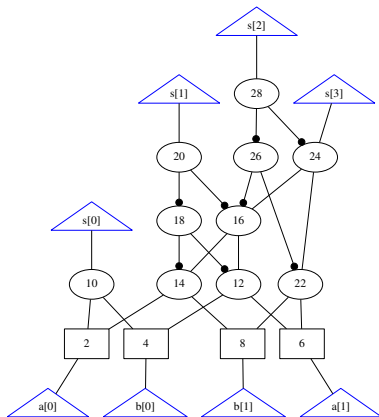
$$-l_{10} + a_0b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0b_1$$

$$-l_{12} + a_1b_0$$

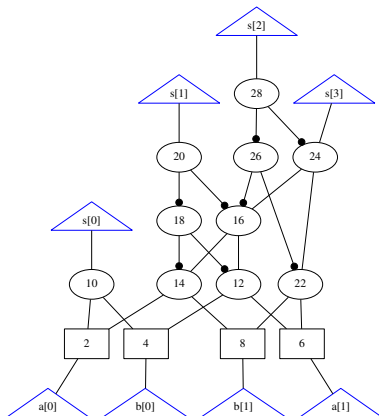
$$-l_{10} + a_0b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

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$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0b_1$$

$$-l_{12} + a_1b_0$$

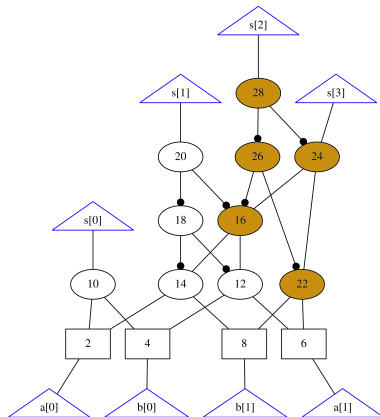
$$-l_{10} + a_0b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

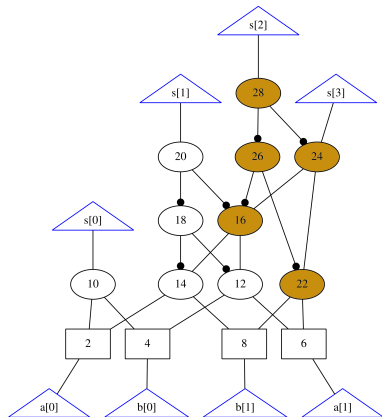
$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
$-s_2 + l_{28}$	$-l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1$
$-s_1 + l_{20}$	$-l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
$-l_{24} + l_{22} l_{16}$	$-l_{10} + a_0 b_0$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



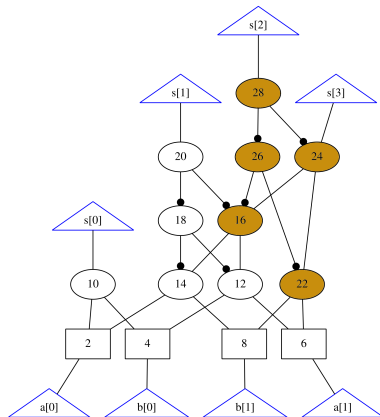
On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
$-s_2 + l_{28}$	$-l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1$
$-s_1 + l_{20}$	$-l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
$-l_{24} + l_{22} l_{16}$	$-l_{10} + a_0 b_0$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$\begin{aligned}
 &8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &- 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4
 \end{aligned}$$



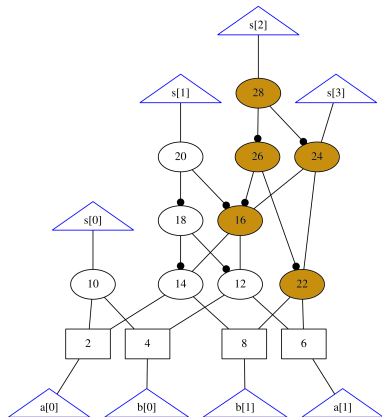
On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
$-s_2 + l_{28}$	$-l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1$
$-s_1 + l_{20}$	$-l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
$-l_{24} + l_{22} l_{16}$	$-l_{10} + a_0 b_0$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$\begin{aligned}
 &8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &- 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4 \\
 &2l_{20} + 4l_{16} - 2l_{14} - 2l_{12}
 \end{aligned}$$



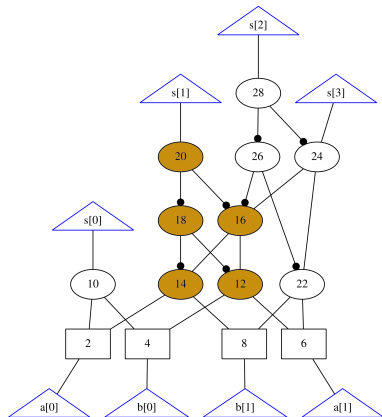
On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
$-s_2 + l_{28}$	$-l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1$
$-s_1 + l_{20}$	$-l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
$-l_{24} + l_{22} l_{16}$	$-l_{10} + a_0 b_0$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$\begin{aligned}
 &8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &- 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4 \\
 &2l_{20} + 4l_{16} - 2l_{14} - 2l_{12}
 \end{aligned}$$



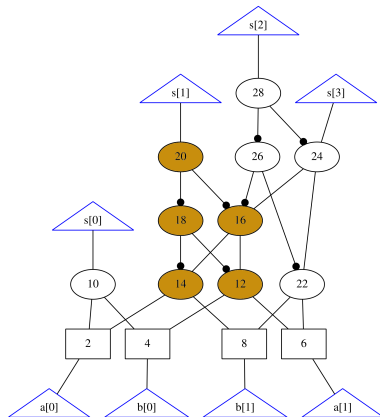
On-the-fly Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$\begin{array}{ll}
 -s_3 + l_{24} & -l_{22} + a_1 b_1 \\
 -s_2 + l_{28} & -l_{20} - l_{18} - l_{16} + 1 \\
 -s_1 + l_{20} & -l_{18} + l_{16} - l_{14} - l_{12} + 1 \\
 -s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
 -l_{28} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
 -l_{26} + l_{24} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\
 -l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0
 \end{array}$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$\begin{aligned}
 &8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &- 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4 \\
 &2l_{20} + 4l_{16} - 2l_{14} - 2l_{12} \\
 &- 2l_{18} + 2l_{16} - 2l_{14} - 2l_{12} + 2
 \end{aligned}$$



On-the-fly Linearization

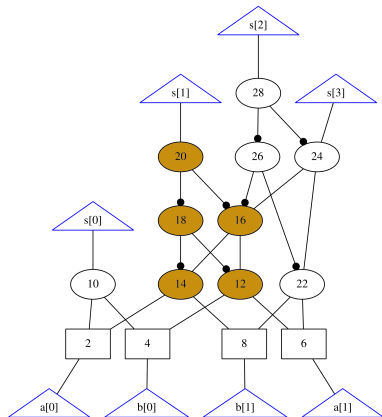
Gate polynomials $G(C) \subseteq \mathbb{Z}[X]$.

$$\begin{array}{ll}
 -s_3 + l_{24} & -l_{22} + a_1 b_1 \\
 -s_2 + l_{28} & -l_{20} - l_{18} - l_{16} + 1 \\
 -s_1 + l_{20} & -l_{18} + l_{16} - l_{14} - l_{12} + 1 \\
 -s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
 -l_{28} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
 -l_{26} + l_{24} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\
 -l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0
 \end{array}$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$\begin{aligned}
 & 8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 & 4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 & 4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 & - 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4 \\
 & 2l_{20} + 4l_{16} - 2l_{14} - 2l_{12} \\
 & - 2l_{18} + 2l_{16} - 2l_{14} - 2l_{12} + 2
 \end{aligned}$$

0



MULTILING

- Builds on AMULET 2.2, written in C++
- Variables are sorted based on minimum distance to primary inputs
- Gröbner basis engine: MSOLVE¹
- Sub-Circuits are iteratively enlarged by increasing the depth
- Non-linear rewriting as fall-back, when distance is below 6.

¹J. Berthomieu, C. Eder, and M. Safey El Din. msolve: A Library for Solving Polynomial Systems. ISSAC, 2021

Evaluation - Optimized Multipliers

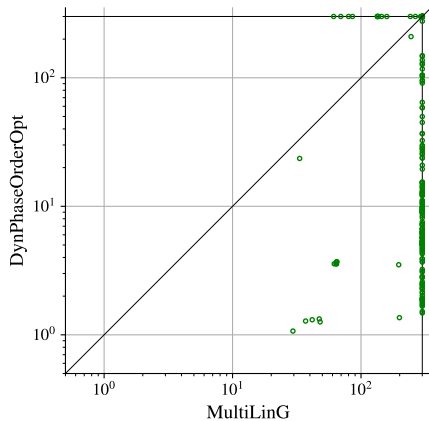
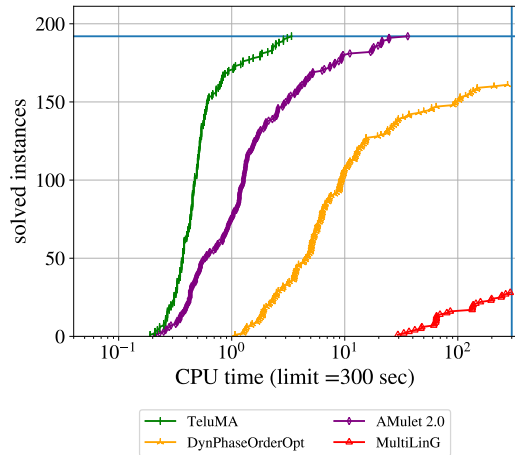
64-bit ABC multipliers

	Related work			Time (s)		Preproc.	MSOLVE Calls						
Synth	TeluMA	AMulet2	DPOO	Total	MSOLVE	MergedN	#	$d = 3$ (s)		$d = 4$ (s)		$d = 5$ (s)	
resyn	0.3	TO	1.0	11.0	0.2	7996	7	4	0.03	2	0.03	1	0.04
resyn2	0.2	TO	1.0	11.7	0.2	7996	6	4	0.03	1	0.03	1	0.04
resyn3	0.3	0.2	1.0	11.2	-	8000	0	0		0		0	
dc2	0.2	0.3	1.1	11.1	-	8000	0	0		0		0	
comp	EE	TO	1.0	12.2	0.2	7996	6	4	0.03	1	0.03	1	0.04

time in sec, TO = 300 sec, EE = error, DPOO = DYNPHASEORDEROPT

Evaluation - 64-bit Multipliers

Verification of 192 unsigned 64-bit multipliers



Conclusion

**If the specification polynomial is linear,
a Gröbner basis with respect to a
degree reverse lexicographic term ordering
contains linear polynomials that suffice
to derive correctness of the circuit.**

- Full Gröbner basis computation is hard
- Our approach linearizes polynomials on the fly
- Robust on optimized benchmarks and complements existing $\prec_{\text{lex}}$ techniques.

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