VERIFYING ARITHMETIC CIRCUITS WITH (LINEAR) POLYNOMIALS

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Invited Talk
SYNASC & AVM, Timisoara, Romania

September 24, 2025

TU WIEN Informatics
FWF Austrian Science Fund

$$4\,195\,835 \div 3\,145\,727 = ?$$

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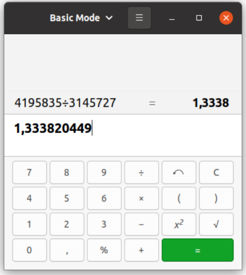

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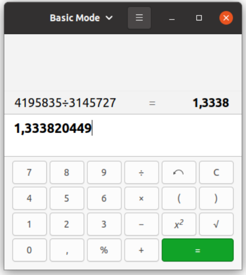

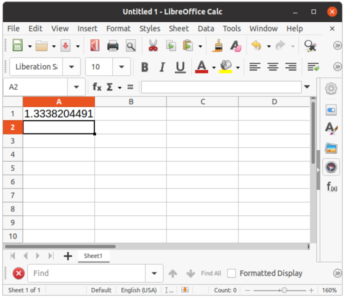

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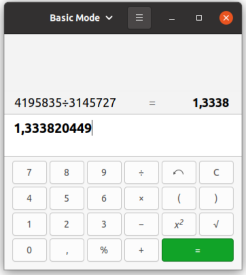

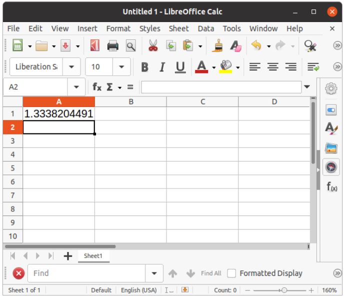

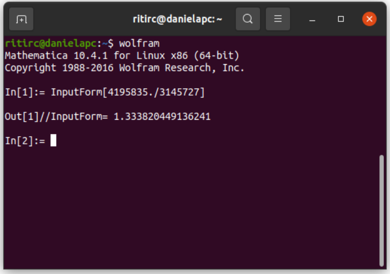

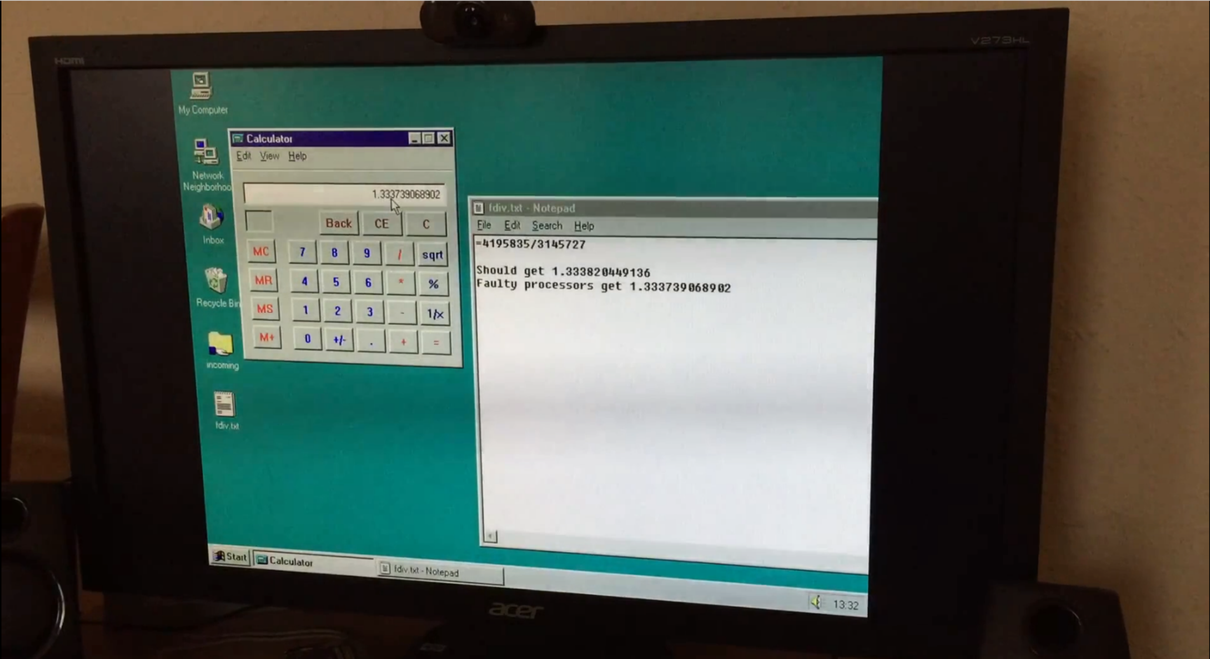

Pentium FDIV bug and AMI GUI BIOS demo Quelle: https://youtu.be/hE7qMJVl15U

Intel Pentium FDIV-Bug 1994

Quelle: http://neology.com.au/portfolios/a80502-90-sx923/

- Affected floating point unit (FPU) in early Intel processors.
- Processor might return incorrect result for division.
- Cost in 1994: 500 million dollars.

Even more than 30 years later verification of arithmetic circuits is considered to be hard. Correctness proofs are not fully automated yet.

Challenge: Integer multipliers

Integer Multiplication

$$\begin{array}{ccccc} 1 & 1 & \cdot & 1 & 1 \\ \hline & & & 1 & 1 \\ & & 1 & 1 & \\ & {}_{1} & {}_{1} & {}_{0} & \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

$3 \cdot 3 = 9$

Integer Multiplication

$$
\begin{array}{cccccc}
 & & 1 & 1 & \cdot & 1 \; 1 \\
\hline
 & & & & 1 & 1 \\
 & & {}_1 1 & {}_1 1 & {}_0 & \\
\hline
 & 1 & 0 & 0 & 1 &
\end{array}
$$

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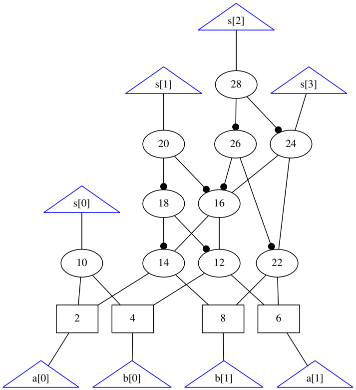


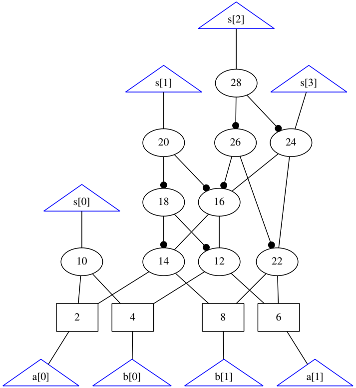
And-Inverter Graph

Multiplier Circuits

Given: Gate-level multiplier for fixed bit-width.

Question: For all possible $a_i, b_i \in \mathbb{B}$:

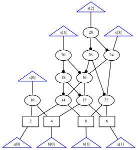
$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$



And-Inverter Graph

Formal Verification

System



Specification

For all $a_i, b_i \in \mathbb{B}$:

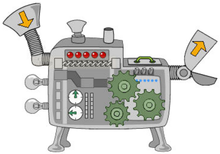
$$(2a_1 + a_0) * (2b_1 + b_0)$$

$$= 8s_3 + 4s_2 + 2s_1 + s_0 ?$$

Mathematical Model

$$
\begin{aligned}
B = \{ \\
& x - a_0 * b_0, \\
& y - a_1 * b_1, \\
& s_0 - x * y, \\
& \dots \\
\}
\end{aligned}
$$

Automated Decision Process



✓ | ✗

Formal Verification Techniques

Decision Diagrams

- First technique to detect Pentium bug

  [ChenBryant DAC'95]

- Requires knowledge of the layout

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Theorem Proving

- Used in industry, e.g., ACL2

  [TemelSlobodovaHunt CAV'20]

- Automated on the RTL level

  [Temel TACAS'24]

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Satisfiability Checking (SAT)

- SAT 2016: Exponential run-time of solvers [Biere SATComp'16]
- SAT 2024: Equivalence checking of structural similar circuits
  [BiereFallerFazekasFleuryFroleyksPollitt SATComp'24]

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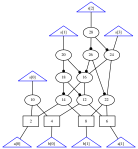
Algebraic Approach

- Seminal work: [LvKallaEnescu TCAD'13, CiesielskiYuBrownLiuRossi DAC'15] [SayedGroßeKühneSoekenDrechsler DATE'16]
- Polynomial encoding
- Works for non-trivial multiplier designs

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$$B = \{ \\ x - a_0 * b_0, \\ y - a_1 * b_1, \\ s_0 - x * y, \\ \dots \\ \}$$

Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right)$$

Implication

$= 0$ ✓

$\neq 0$ ✗

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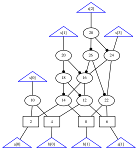

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Ideal Membership

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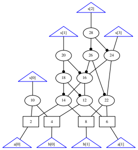

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Multiplier Specification

Unsigned Integers:

$$0 = \sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right) \quad \in \mathbb{Q}[X]$$

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Signed Integers:

$$-2^{2n-1} s_{2n-1} + \sum_{i=0}^{2n-2} 2^i s_i - \left(-2^{n-1} a_{n-1} + \sum_{i=0}^{n-2} 2^i a_i\right)\left(-2^{n-1} b_{n-1} + \sum_{i=0}^{n-2} 2^i b_i\right) \quad \in \mathbb{Q}[X]$$

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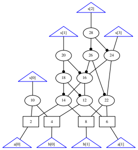

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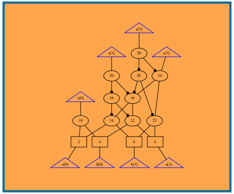

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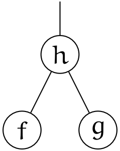
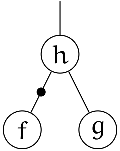
From AIGs to Polynomials



$h = f \wedge g$



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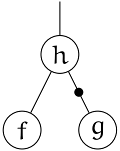

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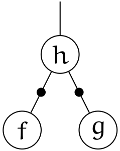

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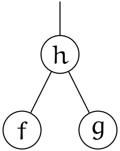
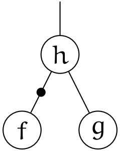


$h = f \wedge g$

f	g	h
0	0	0
0	1	0
1	0	0
1	1	1



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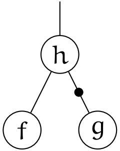

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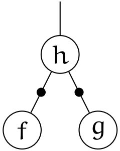

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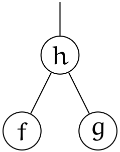

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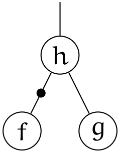
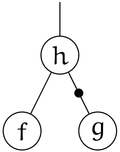
$$-h + fg$$



$$h = \neg f \wedge g$$

f	g	h
0	0	0
0	1	1
1	0	0
1	1	0

$$-h - fg + g$$



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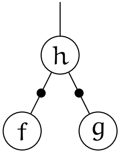

$$h = \neg f \wedge \neg g$$

f	g	h
0	0	1
0	1	0
1	0	0
1	1	0

$$-h + fg - f - g + 1$$

From AIGs to Polynomials

Gate polynomials $G(C) \subseteq \mathbb{Q}[X]$.

$$
\begin{array}{ll}
-s_3 + l_{24} & -l_{22} + a_1 b_1 \\
-s_2 + l_{28} & -l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1 \\
-s_1 + l_{20} & -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\
-s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
-l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
-l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\
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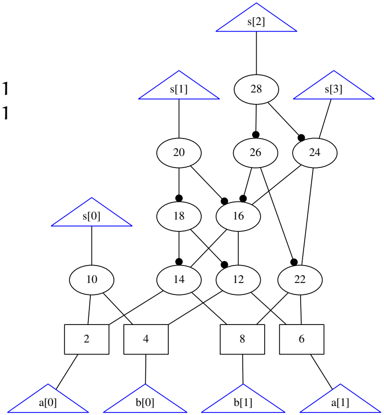

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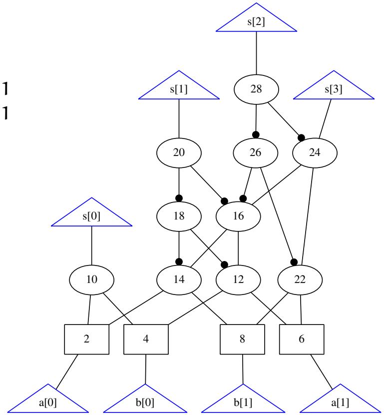
\end{array}
$$

Boolean input constraints $B(C) \subseteq \mathbb{Q}[X]$.

$$
\begin{array}{ll}
a_1, a_0 \in \mathbb{B} & -a_1^2 + a_1,\ -a_0^2 + a_0, \\
b_1, b_0 \in \mathbb{B} & -b_1^2 + b_1,\ -b_0^2 + b_0
\end{array}
$$

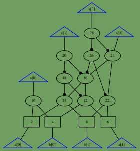
Specification $\mathcal{S}_n \in \mathbb{Q}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1 a_1 - 2b_1 a_0 - 2b_0 a_1 - b_0 a_0$$



Basic Idea of Algebraic Approach

Multiplier



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Ideal Membership

$= 0$ ✓

$\neq 0$ ✗

COMPUTER ALGEBRA

COMPUTER ALGEBRA

IDEALS AND GRÖBNER BASES

Ideal

Ideal. A subset $I \subset \mathbb{K}[X]$ is an ideal if it satisfies:

- $0 \in I$
- If $f, g \in I$, then $f + g \in I$.
- If $f \in I$ and $h \in \mathbb{K}[X]$ then $hf \in I$.

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Ideal generated by a finite number of polynomials.
Let $f_1, \dots, f_s \in \mathbb{K}[X]$. Then we set

$$\langle f_1, \dots, f_s \rangle = \{h_1 f_1 + \cdots + h_s f_s \mid h_1, \dots, h_s \in \mathbb{K}[X]\}.$$

$\langle f_1, \dots, f_s \rangle$ is an ideal and is called the ideal generated by $f_1, \dots, f_s$.

We can think of $\langle f_1, \dots, f_s \rangle$ as consisting of all “polynomial consequences” of the equations $f_1 = f_2 = \ldots = f_s = 0$.

Applications of Ideals

The Ideal Membership Problem.

Given $f \in \mathbb{K}[X]$ and an ideal $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[X]$, determine if $f \in I$.

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The Ideal Membership Problem.

Given $f \in \mathbb{K}[X]$ and an ideal $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[X]$, determine if $f \in I$.

Divide f by $f_1, \ldots, f_s$?

Orderings on the Monomials

Univariate Polynomials - Sort by Degree.

$$\cdots > x^{m+1} > x^m > \cdots > x^2 > x > 1$$

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Linear Polynomials - Sort by variables.

$$x_1 > x_2 > \cdots > x_n$$

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Univariate Polynomials - Sort by Degree.

$$\cdots > x^{m+1} > x^{m} > \cdots > x^{2} > x > 1$$

Linear Polynomials - Sort by variables.

$$x_1 > x_2 > \cdots > x_n$$

How to order non-linear multivariate polynomials in $\mathbb{K}[X]$?

Orderings on the Monomials

Let $\sigma_1 = x_1^{u_1} \cdots x_n^{u_n}, \qquad \sigma_2 = x_1^{v_1} \cdots x_n^{v_n}.$

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Lexicographic Order $\prec_{\text{lex}}$.

$\sigma_1 \prec_{\text{lex}} \sigma_2$ iff there exists an index i with $u_j = v_j$ for all $j < i$, and $u_i < v_i$.

Example: $x_1^2 x_2^7 \prec_{\text{lex}} x_1^3 x_2 \prec_{\text{lex}} x_1^3 x_2^6$

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Degree Lexicographic Order $\prec_{\text{dlex}}$.

$\sigma_1 \prec_{\text{dlex}} \sigma_2$ iff either $|\sigma_1| < |\sigma_2|$ or $|\sigma_1| = |\sigma_2|$ and $\sigma_1 \prec_{\text{lex}} \sigma_2$.

Example: $x_1^3 x_2 \prec_{\text{dlex}} x_1^2 x_2^7 \prec_{\text{dlex}} x_1^3 x_2^6$

Gröbner Bases

A Gröbner basis is a special kind of generating set for an ideal with the following properties:

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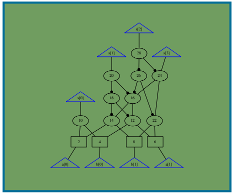
4. Given an arbitrary basis, we can compute G (double-exponential).

BACK TO CIRCUITS

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$$B = \{\; x - a_0 * b_0,\; y - a_1 * b_1,\; s_0 - x * y,\; \dots \;\}$$

Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right)$$

Ideal Membership

$= 0$ ✓

$\neq 0$ ✗

Ideal Membership Problem

- Polynomial Encoding:
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Multiplier. A circuit C is called a multiplier if

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right) \quad \in \quad J(C).$$

Verification Algorithm

Theorem (Kaufmann, Biere, Kauers FMCAD'17) For any circuit C, the set $G(C) \cup B(C)$ forms a Gröbner basis for $J(C)$ w.r.t. any lexicographic monomial order where "gate inputs $<$ gate output".

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Reduce specification $\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right)$ by elements of $G(C) \cup B(C)$ until no further reduction is possible, then C is a multiplier iff remainder is zero.

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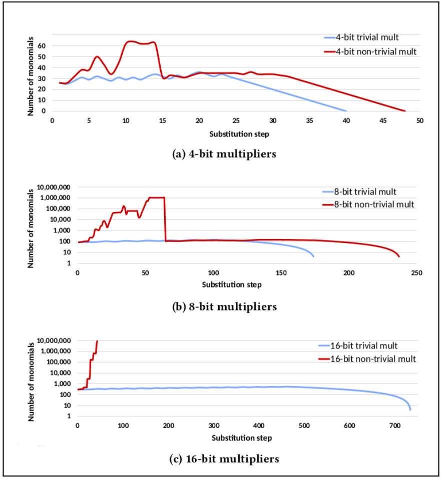
Computational Problems

- The number of monomials in the intermediate results blows-up.
- 8-bit multiplier cannot be verified within 20 minutes.

Verification Algorithm

[MahzoonGroßeDrechsler ICCAD'18]



Strategies

1. Encoding
   - Embedding different phases [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]
2. Preprocessing
   - Variable Elimination [MahzoonGroßeDrechsler DAC'19, RitircBiereKauers DATE'18]
3. Reduction
   - Incremental Algorithm [RitircBiereKauers FMCAD'17]
   - Dynamic Reduction Order [MahzoonGroßeSchollDrechsler DATE'20, KonradScholl FMCAD'24]
4. Tricky: OR Gates in final stage adder
   - Include SAT or BDDs [KaufmannBiereKauers FMCAD'19, DrechslerMahzoon ISEEIE'22]

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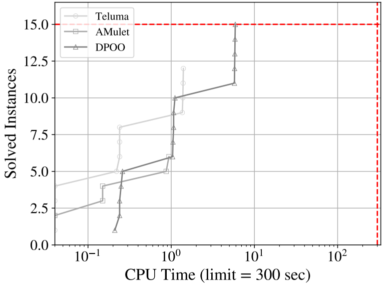
All of these strategies rely on a **lexicographic term ordering** and aim to find an optimal reduction order.

Strategies

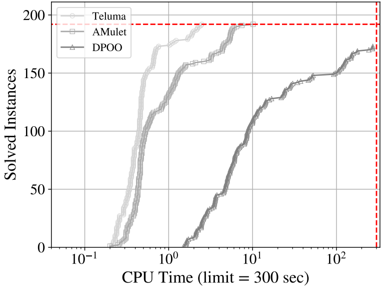
Optimized multipliers

- 5 architectures
- for 32, 64, 128 bit inputs



Unoptimized multipliers

- 192 architectures
- 64 bit inputs



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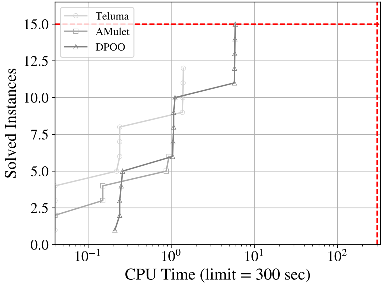

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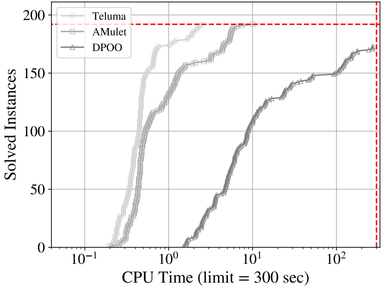

👎 Fast approaches are overfitted

A New Idea

Change monomial order to degree-lexicographic order (Kaufmann, Berthomieu TACAS'25)

	$\prec_{\text{lex}}$	$\prec_{\text{dlex}}$
GB Computation	Easy	Hard
Spec Reduction	Hard	Easy

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GB Computation	Easy	Hard
Spec Reduction	Hard	Easy

Theorem (Kaufmann, Berthomieu TACAS'25) If the specification polynomial is linear, then the linear polynomials in a $\prec_{\text{dlex}}$-GB suffice to verify the circuit.

Verification via Linearization

Input: Circuit C, specification polynomial S
Output: Determine whether C fulfils S

$G \leftarrow G(C) \cup B(C)$

$S_{\mathrm{lin}}, G \leftarrow$ Linearize-Spec(S, G)

$L \leftarrow$ LINEAR-POLIES-OF-$\prec_{\mathrm{dlex}}$-GB(G)

$S_{\mathrm{lin}} \leftarrow$ Linear-Reduce(S_{lin}, L)

return $S_{\mathrm{lin}} \stackrel{?}{=} 0$

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$S_{\mathrm{lin}}, G_{\mathrm{ext}} \leftarrow$ Linearize-Spec(S, G)

Gate polynomials $G(C) \subseteq \mathbb{Q}[X]$.

$$
\begin{aligned}
&-s_3 + l_{24} && -l_{22} + a_1 b_1 \\
&-s_2 + l_{28} && -l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1 \\
&-s_1 + l_{20} && -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\
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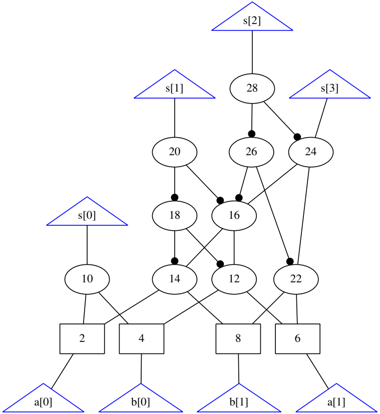

Specification $S \in \mathbb{Q}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1 a_1 - 2b_1 a_0 - 2b_0 a_1 - b_0 a_0$$

$S_{\text{lin}}, G_{\text{ext}} \leftarrow$ **Linearize-Spec**(S, G)

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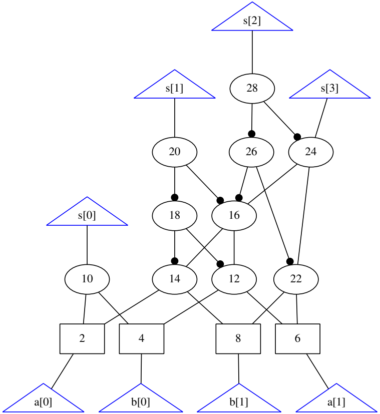
\end{array}
$$

Extension polynomials $G_{\text{ext}} \subseteq \mathbb{Q}[X]$.

$$
\begin{array}{ll}
-t_{11} + a_1 b_1 & -t_{01} + a_0 b_1 \\
-t_{10} + a_1 b_0 & -t_{00} + a_0 b_0
\end{array}
$$

Linear Specification $S_{\text{lin}} \in \mathbb{Q}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4t_{11} - 2t_{10} - 2t_{01} - t_{00}$$



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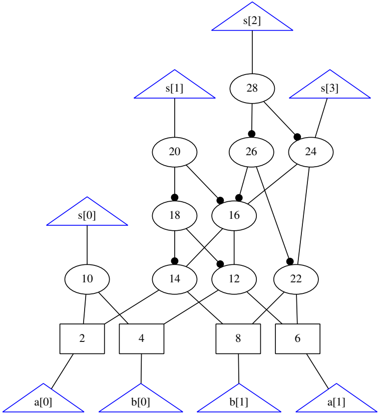

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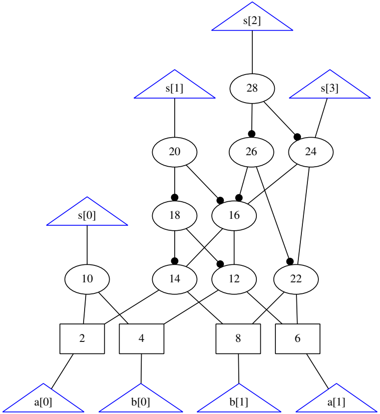

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Computing a $\prec_{\text{dlex}}$-GB for the whole circuit is infeasible.

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5 **return** $S_{\mathrm{lin}} \stackrel{?}{=} 0$

compute $\prec_{\mathrm{dlex}}$-GB of $G' \subseteq G$

Computing a $\prec_{\mathrm{dlex}}$-GB for the whole circuit is infeasible.

$\rightsquigarrow$ work locally: extract sub-circuits and only compute $\prec_{\mathrm{dlex}}$-GBs for them

Local Linearization

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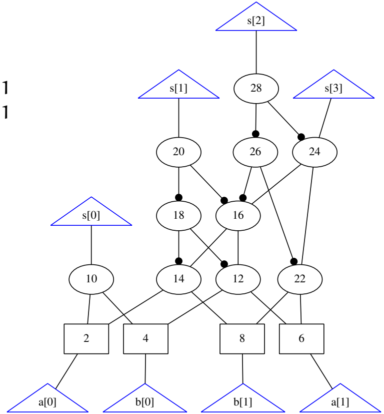

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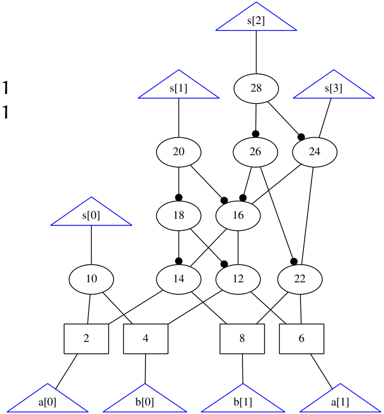

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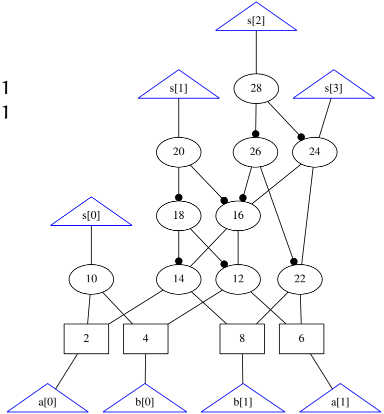

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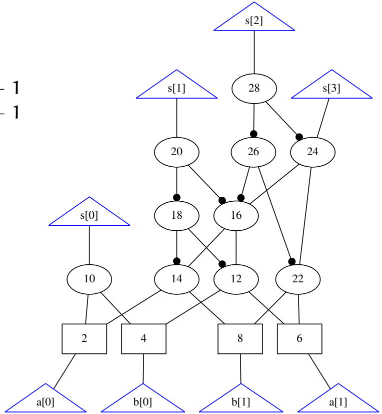

Local Linearization

Gate polynomials $G(C) \subseteq \mathbb{Q}[X]$.

$$
\begin{array}{ll}
-s_3 + l_{24} & -l_{22} + a_1 b_1 \\
-s_2 + l_{28} & -l_{20} + l_{18} l_{16} - l_{18} - l_{16} + \\
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-s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
\boxed{-l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1} & -l_{14} + a_0 b_1 \\
-l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\
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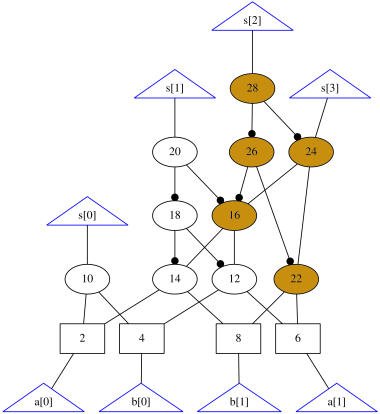
\end{array}
$$



Specification $\mathcal{S}_n \in \mathbb{Q}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

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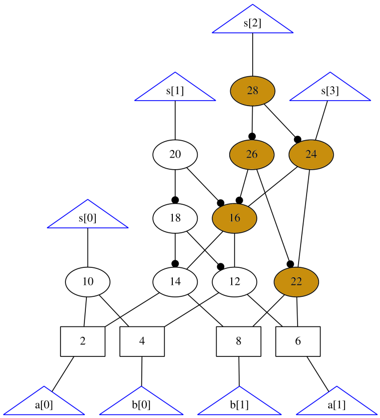

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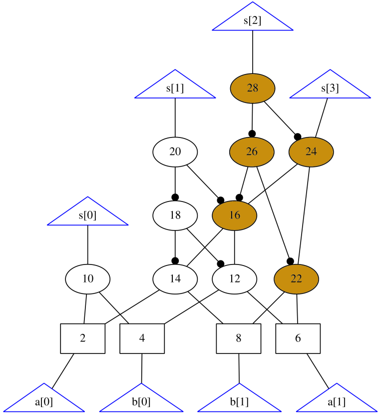
\end{aligned}
$$



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2l_{20} + 4l_{16} - 2l_{14} - 2l_{12}
\end{aligned}
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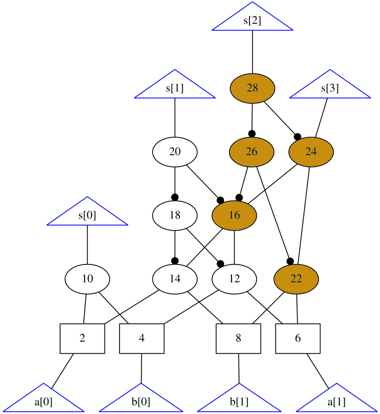

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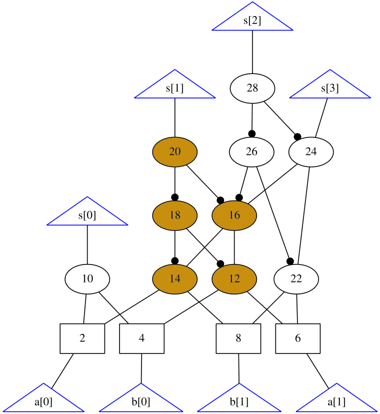
\end{aligned}
$$



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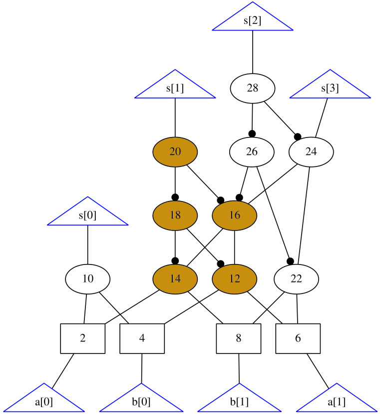
$$2l_{20} + 4l_{16} - 2l_{14} - 2l_{12}$$

$$-2l_{18} + 2l_{16} - 2l_{14} - 2l_{12} + 2$$



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$$
\begin{aligned}
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&-l_{24} + l_{22} l_{16}
\end{aligned}
$$

$$
\begin{aligned}
&-l_{22} + a_1 b_1 \\
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&-l_{18} + l_{16} - l_{14} - l_{12} + 1 \\
&-l_{16} + l_{14} l_{12} \\
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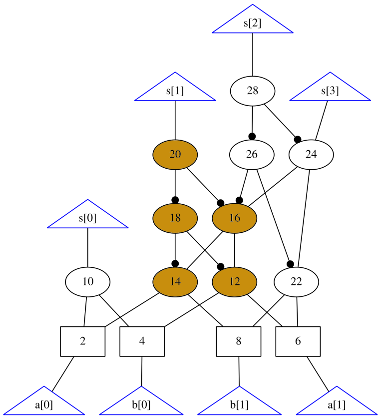
0
\end{aligned}
$$

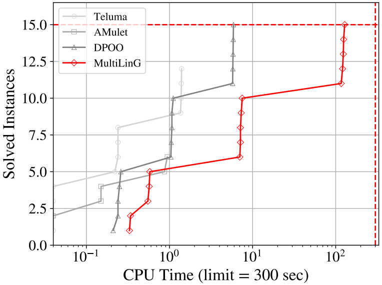


First Approach

(Kaufmann, Berthomieu TACAS'25)

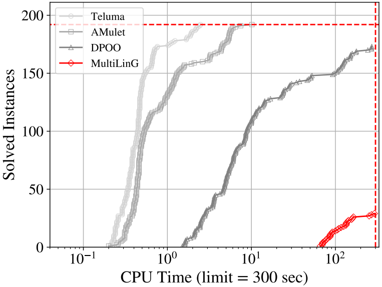
Optimized multipliers

- 5 architectures
- for 32, 64, 128 bit inputs



Unoptimized multipliers

- 192 architectures
- 64 bit inputs



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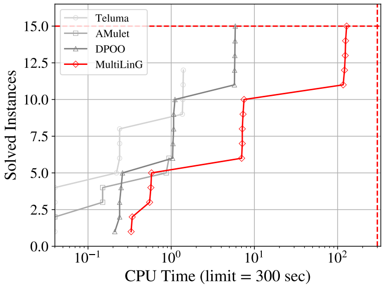

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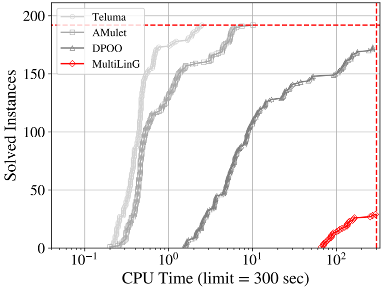

- Approach seems to be more robust against optimization
- Black-box GB computation is too costly (can handle ≤ 8 vars)
- Completely ignores existing structure

Two Targeted Approaches

(Hofstadler, Kaufmann, CP'25)

FGLM-style Linearization

Guess & Prove Linearization

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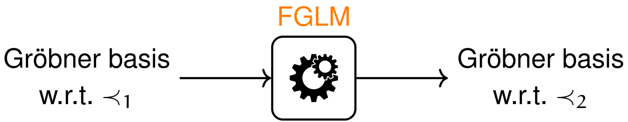


One can adapt FGLM to compute all linear polynomials and then stop.

Exploit existing structure $\rightsquigarrow$ a lot faster than GB computation

FGLM-style Linearization

Exploit that we already have ($\prec_{\text{lex}}$-)Gröbner basis.

Gröbner basis w.r.t. $\prec_1$ $\xrightarrow{\text{FGLM}}$ Gröbner basis w.r.t. $\prec_2$

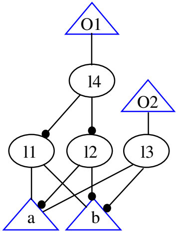
One can adapt FGLM to compute all linear polynomials and then stop.

- Exploit existing structure $\rightsquigarrow$ a lot faster than GB computation
- Runtime exponential in number of variables (can handle ≤ 15 vars)

FGLM-style Linearization

Ideal $I \subseteq \mathbb{Q}[a, b, l_1, l_2, l_3, l_4]$ **generated by:**

$$l_1 - ab, \quad l_2 - (1-a)(1-b), \quad l_3 - a(1-b), \quad l_4 - (1-l_1)(1-l_2), \quad a^2 - a, \quad b^2 - b$$



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1. Compute Normalforms:

$$l_1 = ab, \quad l_2 = ab - a - b + 1, \quad l_3 = -ab + a, \quad l_4 = -2ab + a + b$$

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2. Build Coefficient Matrix A:

$$\begin{array}{cc} \begin{array}{ccccccc} NF_G(1) & NF_G(a) & NF_G(b) & NF_G(l_1) & NF_G(l_2) & NF_G(l_3) & NF_G(l_4) \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -2 \end{pmatrix} & \begin{array}{c} 1 \\ a \\ b \\ ab \end{array} \end{array}$$

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3. Compute Kernel of A:

$$\begin{array}{c} \begin{array}{ccccccc} 1 & a & b & l_1 & l_2 & l_3 & l_4 \end{array} \\ \begin{pmatrix} 0 & -1 & -1 & 2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

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4. Read off Linear Polynomials:

$$l_4 + 2l_1 - a - b, \qquad l_3 + l_1 - a, \qquad l_2 - l_1 + a + b - 1$$

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- Exploit that we can compute (a lot of) points in the variety $V(G)$

Guess & Prove Linearization

Exploit that we can compute (a lot of) points in the variety $V(G)$

...and that our ideals are radical
...and that our ideals are zero-dimensional
...and that our ideals don't have additional roots in algebraic closure
...and that we can efficiently test ideal membership of linear polynomials

Guess & Prove Linearization

Given: $G \subseteq \mathbb{K}[X]$ satisfying the assumptions from before

Compute: $\mathbb{K}$-VS basis of $\{\, f \in \langle G \rangle \mid \deg(f) \leq 1 \,\}$

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Algorithm

```
Sample points P ⊆ V(G)
Compute all linear polynomials F that vanish on P
if F ⊆ ⟨G⟩
  return F
else add witness points to P and repeat
```

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Algorithm

1. Sample points $P \subseteq V(G)$ "guess"
2. Compute all linear polynomials F that vanish on P
3. **if** $F \subseteq \langle G \rangle$
4. **return** F
5. **else** add witness points to P and repeat

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Algorithm

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Compute all linear polynomials F that vanish on P
if $F \subseteq \langle G \rangle$ "prove"
  return F
else add witness points to P and repeat

Guess & Prove Linearization

Given: $G \subseteq \mathbb{K}[X]$ satisfying the assumptions from before

Compute: $\mathbb{K}$-VS basis of $\{ f \in \langle G \rangle \mid \deg(f) \leq 1 \}$

Algorithm

Sample points $P \subseteq V(G)$ "guess"
Compute all linear polynomials F that vanish on P
if $F \subseteq \langle G \rangle$ "prove"
  return F "repair"
else add witness points to P and repeat

Guess & Prove Linearization

Let $G \subseteq \mathbb{Q}[a, b, l_1, l_2, l_3, l_4]$ **consist of:**

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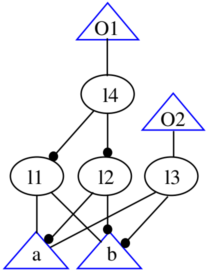
Sample: $(0, 0, 0, 1, 0, 0),\ (1, 0, 0, 0, 1, 1),\ (0, 1, 0, 0, 0, 1) \in V(G)$



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Sample: $(0, 0, 0, 1, 0, 0),\ (1, 0, 0, 0, 1, 1),\ (0, 1, 0, 0, 0, 1) \in V(G)$

Matrix A**:**

$$\begin{array}{c} \begin{array}{ccccccc} 1 & a & b & l_1 & l_2 & l_3 & l_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

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Kernel A**:**

$$\begin{array}{c} \begin{array}{ccccccc} 1 & a & b & l_1 & l_2 & l_3 & l_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Guess & Prove Linearization

Let $G \subseteq \mathbb{Q}[a, b, l_1, l_2, l_3, l_4]$ **consist of:**

$$l_1 - ab, \quad l_2 - (1-a)(1-b), \quad l_3 - a(1-b), \quad l_4 - (1-l_1)(1-l_2), \quad a^2 - a, \quad b^2 - b$$

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Kernel yields Guesses:

$l_4 - b - a \qquad l_3 - a \qquad l_2 + a + b - 1 \qquad l_1$

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Guess: $\quad l_4 - b - a \quad\quad l_3 - a \quad\quad l_2 + a + b - 1 \quad\quad l_1$

Prove:

👎 Algebraic proof (reduction by G) does not scale!

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Guess: $l_4 - b - a \qquad l_3 - a \qquad l_2 + a + b - 1 \qquad l_1$

Prove:

- Algebraic proof (reduction by G) does not scale!
- Use a SAT solver!

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Encode $\displaystyle\bigwedge_{g \in G} (g = 0) \;\wedge\; (\text{guess} \neq 0)$ as propositional formula and give to a SAT solver.

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UNSAT, then guess $\in \langle G \rangle$

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In this case, SAT solver returns SAT and $p = (1, 1, 1, 0, 0, 0)$.

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Repair Samples $\quad (0,0,0,1,0,0),\ (1,0,0,0,1,1),\ (0,1,0,0,0,1),\ (1,1,1,0,0,0) \in V(G)$

Matrix A:

$$\begin{array}{c}
\begin{array}{ccccccc} 1 & a & b & l_1 & l_2 & l_3 & l_4 \end{array} \\
\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}
\end{array}$$

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Kernel A**:**

$$\begin{array}{ccccccc} 1 & a & b & l_1 & l_2 & l_3 & l_4 \\ \end{array}$$
$$\begin{pmatrix} 0 & -1 & -1 & 2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix}$$

Kernel yields:

$$l_4 + 2l_1 - a - b, \qquad l_3 + l_1 - a, \qquad l_2 - l_1 + a + b - 1$$

Guess & Prove Linearization

Works really well also for larger circuits (can handle ≥ 200 variables)!

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…how many iterations of the repair loop do we need?

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…how many samples do we need? $\sim 2^{14} \approx \frac{|V(G)|}{10\,000\,000\,000\,000\,000\,000}$

…how many iterations of the repair loop do we need? ≤ 5

Verification via Linearization

Input: Circuit C, specification polynomial S
Output: Determine whether C fulfils S

$G \leftarrow G(C) \cup B(C)$

$S_{\text{lin}}, G \leftarrow$ Linearize-Spec(S, G)

$L \leftarrow \boxed{\text{LINEAR-POLIES-OF-}\prec_{\text{dlex}}\text{-GB}(G)}$

compute $\prec_{\text{dlex}}$-GB of $G' \subseteq G$

$S_{\text{lin}} \leftarrow$ Linear-Reduce(S_{lin}, L)

return $S_{\text{lin}} \stackrel{?}{=} 0$

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$L \leftarrow \boxed{\text{LINEAR-POLIES-OF-}{\prec_{\text{dlex}}}\text{-GB}(G)}$

$S_{\text{lin}} \leftarrow$ Linear-Reduce(S_{lin}, L)

return $S_{\text{lin}} \stackrel{?}{=} 0$

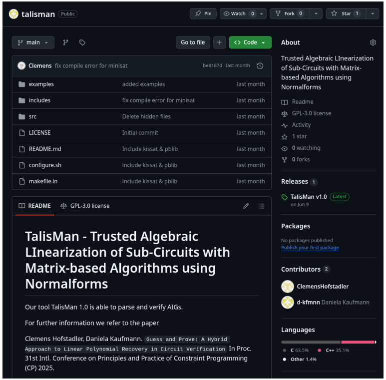
pick $G' \subseteq G$

if $|G'|$ is small use FGLM

else use Guess & Prove

TalisMan

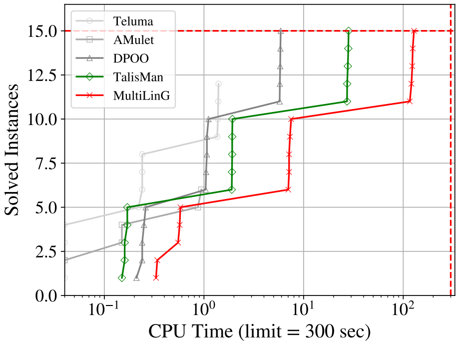
- https://github.com/d-kfmnn/talisman/
- Builds on MultiLing, written in C++
- Uses Flint for matrix computations
- Guess and Prove Algorithm:
  - Uses PBLib to translate guessed polynomial to CNF
  - Uses Kissat as a SAT solver
  - Uses caching of isomorphic subcircuits



Evaluation – Multiplier Circuits

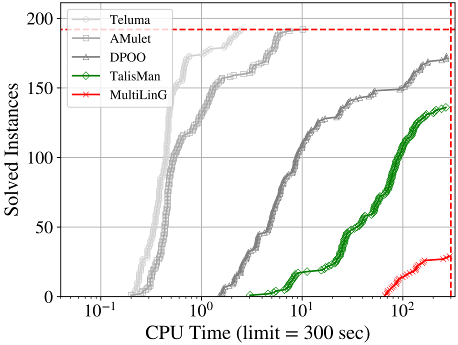
Optimized multipliers

- 5 architectures
- for 32, 64, 128 bit inputs



Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Conclusion

Algebraic circuit verification (GB ✅, Reduction ⚠️)

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💡 Flip things around: $\prec_{\text{lex}} \rightsquigarrow \prec_{\text{dlex}}$ (GB ⚠️, Reduction ✅)

Conclusion

Algebraic circuit verification (GB ✅, Reduction ⚠️)

💡 Flip things around: $\prec_{\text{lex}} \rightsquigarrow \prec_{\text{dlex}}$ (GB ⚠️, Reduction ✅)

Three approaches

Compute GB

- no requirements
- full GB computation
- double exponential runtime
- small circuits (≤ 10 variables)

FGLM

- requires a Gröbner basis
- first steps of FGLM
- exponential runtime
- moderate circuits (≤ 15 variables)

Guess & Prove

- requires variety
- interpolation (guess)
- uses SAT solver (prove)
- larger circuits (≥ 200 variables)

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