COMBINING SAT AND COMPUTER ALGEBRA FOR MULTIPLIER VERIFICATION

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Bugs in hardware are expensive!

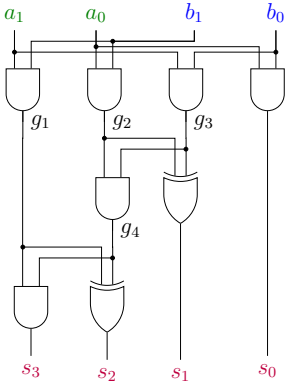
Circuit verification prevents issues like the famous Pentium FDIV bug.

Multiplier verification

Given: Gate-level integer multiplier for fixed bit-width n.

Question: For all possible $a_i, b_i \in \mathbb{B}$:

$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$

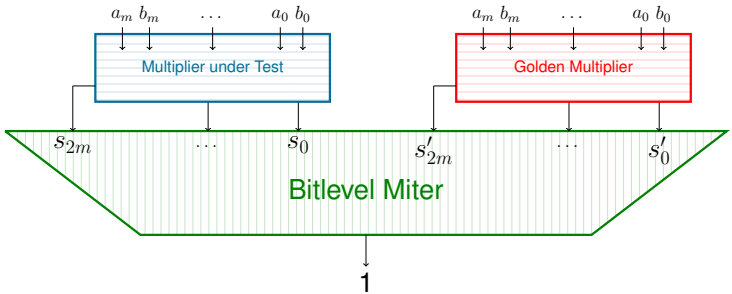


Formal Verification Techniques

Satisfiability Checking (SAT)

- SAT 2016 Competition [Biere SATComp'16]
- Exponential run-time of solvers

Equivalence checking: Miter



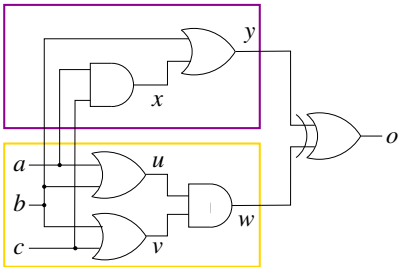
- Miter is translated to CNF.
- CNF is given to SAT solver.
- To show correctness SAT solver needs to return UNSAT.

Tseitin Transformation: Circuit to CNF

$(a \wedge c) \vee b$

=

$(a \vee b) \wedge (b \vee c)$



$$
\begin{aligned}
& o \ \wedge \\
& (x \ \leftrightarrow \ a \wedge c) \wedge \\
& (y \ \leftrightarrow \ b \vee x) \wedge \\
& (u \ \leftrightarrow \ a \vee b) \wedge \\
& (v \ \leftrightarrow \ b \vee c) \wedge \\
& (w \leftrightarrow \ u \wedge v) \wedge \\
& (o \ \leftrightarrow y \oplus w)
\end{aligned}
$$

$$o \wedge (x \rightarrow a) \wedge (x \rightarrow c) \wedge (x \leftarrow a \wedge c) \wedge \ \ldots$$

$$\boxed{o \wedge (\overline{x} \vee a) \wedge (\overline{x} \vee c) \wedge (x \vee \overline{a} \vee \overline{c}) \wedge \ \ldots}$$

Tseitin Transformation: Gate Constraints

Negation:

$$\begin{aligned} x \leftrightarrow \overline{y} &\Leftrightarrow (x \rightarrow \overline{y}) \wedge (\overline{y} \rightarrow x) \\ &\Leftrightarrow (\overline{x} \vee \overline{y}) \wedge (y \vee x) \end{aligned}$$

Disjunction:

$$\begin{aligned} x \leftrightarrow (y \vee z) &\Leftrightarrow (y \rightarrow x) \wedge (z \rightarrow x) \wedge (x \rightarrow (y \vee z)) \\ &\Leftrightarrow (\overline{y} \vee x) \wedge (\overline{z} \vee x) \wedge (\overline{x} \vee y \vee z) \end{aligned}$$

Conjunction:

$$\begin{aligned} x \leftrightarrow (y \wedge z) &\Leftrightarrow (x \rightarrow y) \wedge (x \rightarrow z) \wedge ((y \wedge z) \rightarrow x) \\ &\Leftrightarrow (\overline{x} \vee y) \wedge (\overline{x} \vee z) \wedge (\overline{(y \wedge z)} \vee x) \\ &\Leftrightarrow (\overline{x} \vee y) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z} \vee x) \end{aligned}$$

Equivalence:

$$\begin{aligned} x \leftrightarrow (y \leftrightarrow z) &\Leftrightarrow (x \rightarrow (y \leftrightarrow z)) \wedge ((y \leftrightarrow z) \rightarrow x) \\ &\Leftrightarrow (x \rightarrow ((y \rightarrow z) \wedge (z \rightarrow y)) \wedge ((y \leftrightarrow z) \rightarrow x) \\ &\Leftrightarrow (x \rightarrow (y \rightarrow z)) \wedge (x \rightarrow (z \rightarrow y)) \wedge ((y \leftrightarrow z) \rightarrow x) \\ &\Leftrightarrow (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{z} \vee y) \wedge ((y \leftrightarrow z) \rightarrow x) \\ &\Leftrightarrow (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{z} \vee y) \wedge (((y \wedge z) \vee (\overline{y} \wedge \overline{z})) \rightarrow x) \\ &\Leftrightarrow (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{z} \vee y) \wedge ((y \wedge z) \rightarrow x) \wedge ((\overline{y} \wedge \overline{z}) \rightarrow x) \\ &\Leftrightarrow (\overline{x} \vee \overline{y} \vee z) \wedge (\overline{x} \vee \overline{z} \vee y) \wedge (\overline{y} \vee \overline{z} \vee x) \wedge (y \vee z \vee x) \end{aligned}$$

Checking Commutativity of Multiplication with SAT

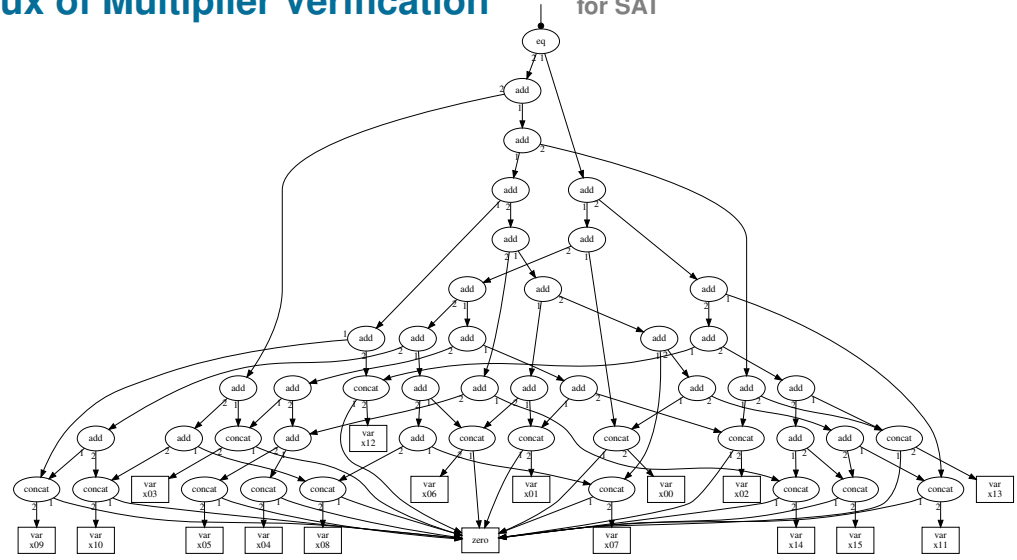
```
bits  Kissat
01      0.00
02      0.00
03      0.00
04      0.00
05      0.00
06      0.02
07      0.12
08      1.23
09      7.85
10     39.42
11    152.89
12    820.12
13    ------

limit of 900 seconds wall clock time
```

```
(set-logic QF_BV)
(declare-fun x () (_ BitVec 12))
(declare-fun y () (_ BitVec 12))
(assert (distinct (bvmul x y) (bvmul y x)))
(check-sat)
```

Crux of Multiplier Verification

for SAT



Formal Verification Techniques

Satisfiability Checking (SAT)

- SAT 2016 Competition
- Exponential run-time of solvers

Decision Diagrams

- First technique to detect Pentium bug
- Rely on manual decomposition

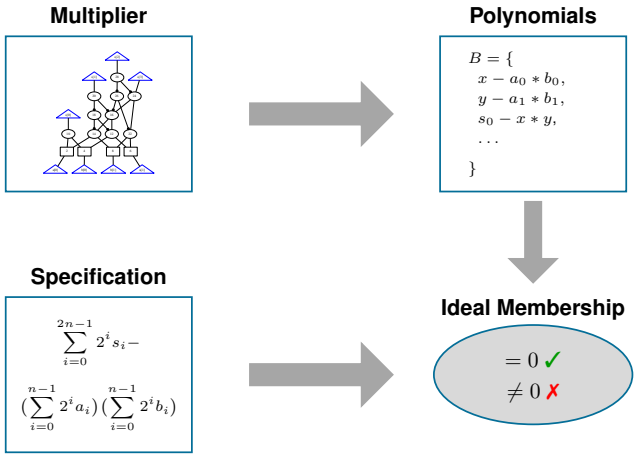
Theorem Proving

- Used in industry
- Requires manual effort
- Automated techniques rely on hierarchical information.

Algebraic Approach

- Great progress since 2015
- Polynomial encoding
- Works for non-trivial multiplier designs

Basic Idea of Algebraic Approach



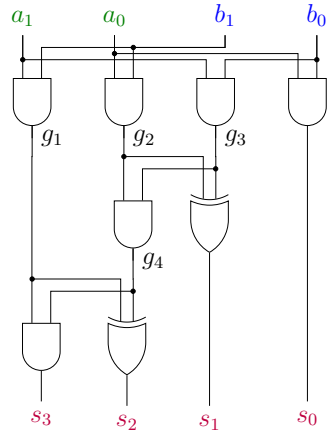
From Circuits to Polynomials

AND-Gate $f \wedge g = y$

f	g	y
0	0	0
0	1	0
1	0	0
1	1	1

XOR-Gate $f \oplus g = y$

f	g	y
0	0	0
0	1	1
1	0	1
1	1	0



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f	g	y
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$$-y + fg$$

XOR-Gate $f \oplus g = y$

f	g	y
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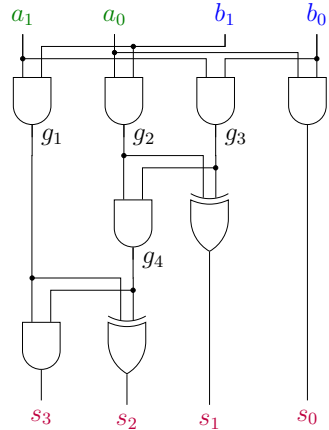
1	0	1
1	1	0

$$-y + f + g - 2fg$$



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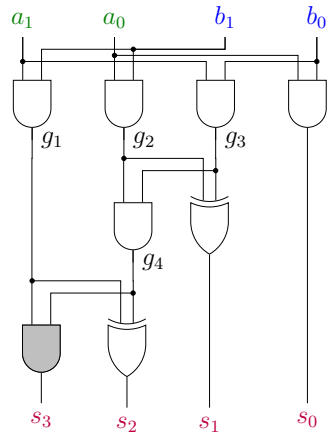
Gate polynomials $G(C) \subset \mathbb{Z}[X]$.

$$s_3 = g_1 \wedge g_4 \qquad -s_3 + g_4 g_1,$$



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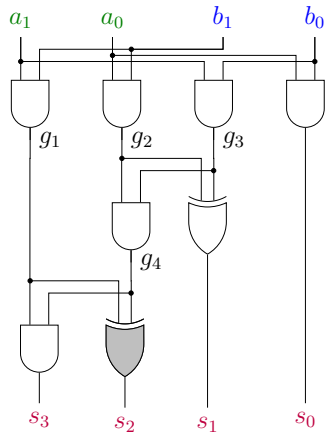
\end{aligned}
$$



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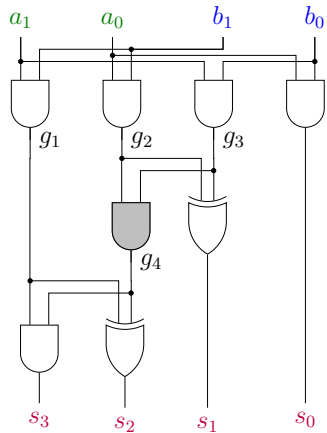

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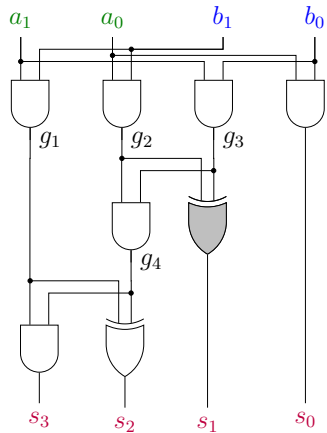

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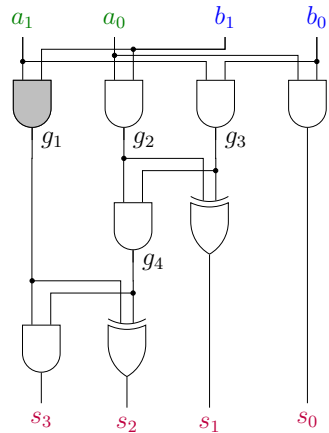

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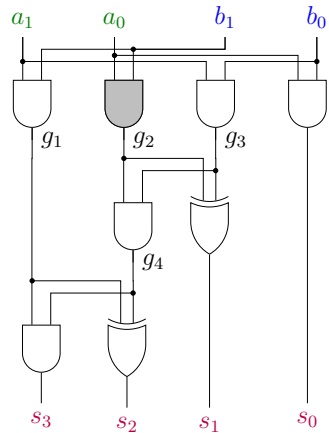

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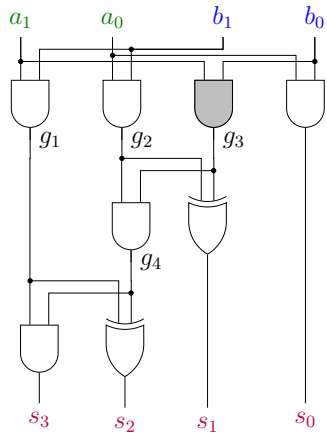

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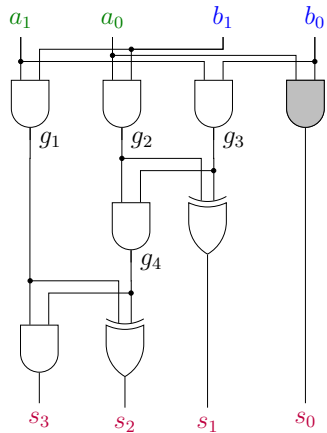
$$



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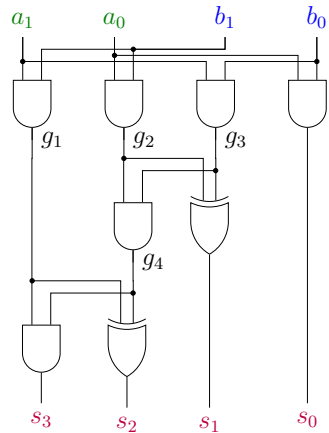
g_4 = g_2 \wedge g_3 & -g_4 + g_2 g_3, \\
s_1 = g_2 \oplus g_3 & -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 = a_1 \wedge b_1 & -g_1 + a_1 b_1, \\
g_2 = a_0 \wedge b_1 & -g_2 + a_0 b_1, \\
g_3 = a_1 \wedge b_0 & -g_3 + a_1 b_0, \\
s_0 = a_0 \wedge b_0 & -s_0 + a_0 b_0
\end{array}
$$

Boolean value constraints $B(C) \subset \mathbb{Z}[X]$.

$$
\begin{array}{ll}
a_1, a_0 \in \mathbb{B} & a_1(1-a_1),\ a_0(1-a_0), \\
b_1, b_0 \in \mathbb{B} & b_1(1-b_1),\ b_0(1-b_0)
\end{array}
$$



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g_4 = g_2 \wedge g_3 \quad & -g_4 + g_2 g_3, \\
s_1 = g_2 \oplus g_3 \quad & -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 = a_1 \wedge b_1 \quad & -g_1 + a_1 b_1, \\
g_2 = a_0 \wedge b_1 \quad & -g_2 + a_0 b_1, \\
g_3 = a_1 \wedge b_0 \quad & -g_3 + a_1 b_0, \\
s_0 = a_0 \wedge b_0 \quad & -s_0 + a_0 b_0
\end{aligned}
$$

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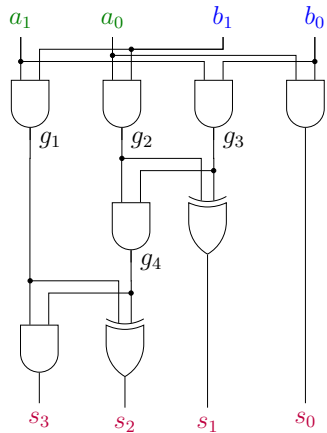
$$
\begin{aligned}
a_1, a_0 \in \mathbb{B} \quad & -a_1^2 + a_1,\ -a_0^2 + a_0, \\
b_1, b_0 \in \mathbb{B} \quad & -b_1^2 + b_1,\ -b_0^2 + b_0
\end{aligned}
$$



Polynomial Encoding

Gate polynomials $G(C) \subset \mathbb{Z}[X]$.

$$
\begin{array}{ll}
s_3 = g_1 \wedge g_4 & -s_3 + g_1 g_4, \\
s_2 = g_1 \oplus g_4 & -s_2 - 2g_1 g_4 + g_4 + g_1, \\
g_4 = g_2 \wedge g_3 & -g_4 + g_2 g_3, \\
s_1 = g_2 \oplus g_3 & -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 = a_1 \wedge b_1 & -g_1 + a_1 b_1, \\
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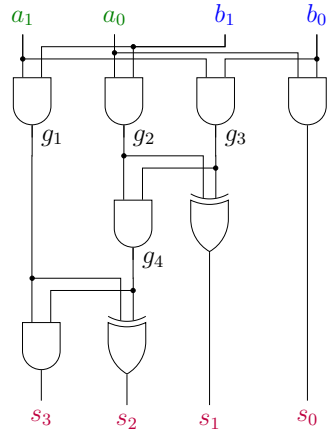
\end{array}
$$

Boolean value constraints $B(C) \subset \mathbb{Z}[X]$.

$$
\begin{array}{ll}
a_1, a_0 \in \mathbb{B} & -a_1^2 + a_1, -a_0^2 + a_0, \\
b_1, b_0 \in \mathbb{B} & -b_1^2 + b_1, -b_0^2 + b_0
\end{array}
$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$



Verification Technique

Gröbner basis

$G(C) \cup B(C)$ is a Gröbner basis w.r.t. the lexicographical ordering.

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Verification Algorithm

Reduce specification $\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right)$ by elements of $G(C) \cup B(C)$ based on a fixed variable order until no further reduction is possible.
Then C is a multiplier iff the final remainder is zero.

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based on a fixed variable order until no further reduction is possible.
Then C is a multiplier iff the final remainder is zero.

Easy: Multipliers containing a ripple-carry adder
Hard: Multipliers containing a generate-and-propagate adder, e.g., carry-lookahead adder

Verification

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&\quad -s_3 + g_1 g_4, \\
&\quad -s_2 - 2g_1 g_4 + g_4 + g_1, \\
&\quad -g_4 + g_2 g_3, \\
&\quad -s_1 - 2g_2 g_3 + g_3 + g_2, \\
&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_1 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
&\quad -b_1^2 + b_1, \\
&\quad -b_0^2 + b_0\}
\end{aligned}
$$

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

Verification

$$
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&-g_2 + a_0 b_1, \\
&-g_3 + a_1 b_0, \\
&-s_0 + a_0 b_0, \\
&-a_1^2 + a_1, \\
&-a_0^2 + a_0, \\
&-b_1^2 + b_1, \\
&-b_0^2 + b_0\}
\end{aligned}
$$

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

Verification

$$
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&\quad -s_1 - 2g_2 g_3 + g_3 + g_2, \\
&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_1 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
&\quad -b_1^2 + b_1, \\
&\quad -b_0^2 + b_0\}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0
\end{aligned}
$$

Verification

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&\quad -s_3 + g_1 g_4, \\
&\quad -s_2 - 2g_1 g_4 + g_4 + g_1, \\
&\quad -g_4 + g_2 g_3, \\
&\quad -s_1 - 2g_2 g_3 + g_3 + g_2, \\
&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_1 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
&\quad -b_1^2 + b_1, \\
&\quad -b_0^2 + b_0 \}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0
\end{aligned}
$$

Verification

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&-s_3 + g_1 g_4, \\
&-s_2 - 2g_1 g_4 + g_4 + g_1, \\
&-g_4 + g_2 g_3, \\
&-s_1 - 2g_2 g_3 + g_3 + g_2, \\
&-g_1 + a_1 b_1, \\
&-g_2 + a_0 b_1, \\
&-g_3 + a_1 b_0, \\
&-s_0 + a_0 b_0, \\
&-a_1^2 + a_1, \\
&-a_0^2 + a_0, \\
&-b_1^2 + b_1, \\
&-b_0^2 + b_0\}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0
\end{aligned}
$$

Verification

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&-s_3 + g_1 g_4, \\
&-s_2 - 2g_1 g_4 + g_4 + g_1, \\
&-g_4 + g_2 g_3, \\
&-s_1 - 2g_2 g_3 + g_3 + g_2, \\
&-g_1 + a_1 b_1, \\
&-g_2 + a_0 b_1, \\
&-g_3 + a_1 b_0, \\
&-s_0 + a_0 b_0, \\
&-a_1^2 + a_1, \\
&-a_0^2 + a_0, \\
&-b_1^2 + b_1, \\
&-b_0^2 + b_0\}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0
\end{aligned}
$$

Verification

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&-s_3 + g_1 g_4, \\
&-s_2 - 2g_1 g_4 + g_4 + g_1, \\
&-g_4 + g_2 g_3, \\
&-s_1 - 2g_2 g_3 + g_3 + g_2, \\
&-g_1 + a_1 b_1, \\
&-g_2 + a_0 b_1, \\
&-g_3 + a_1 b_0, \\
&-s_0 + a_0 b_0, \\
&-a_1^2 + a_1, \\
&-a_0^2 + a_0, \\
&-b_1^2 + b_1, \\
&-b_0^2 + b_0\}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + s_0 - 2a_1 b_0 - a_0 b_0
\end{aligned}
$$

Verification

$$
\begin{aligned}
G(C) \cup B(C) = \{ & \\
& -s_3 + g_1 g_4, \\
& -s_2 - 2g_1 g_4 + g_4 + g_1, \\
& -g_4 + g_2 g_3, \\
& -s_1 - 2g_2 g_3 + g_3 + g_2, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_1 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
& -b_1^2 + b_1, \\
& -b_0^2 + b_0 \}
\end{aligned}
$$

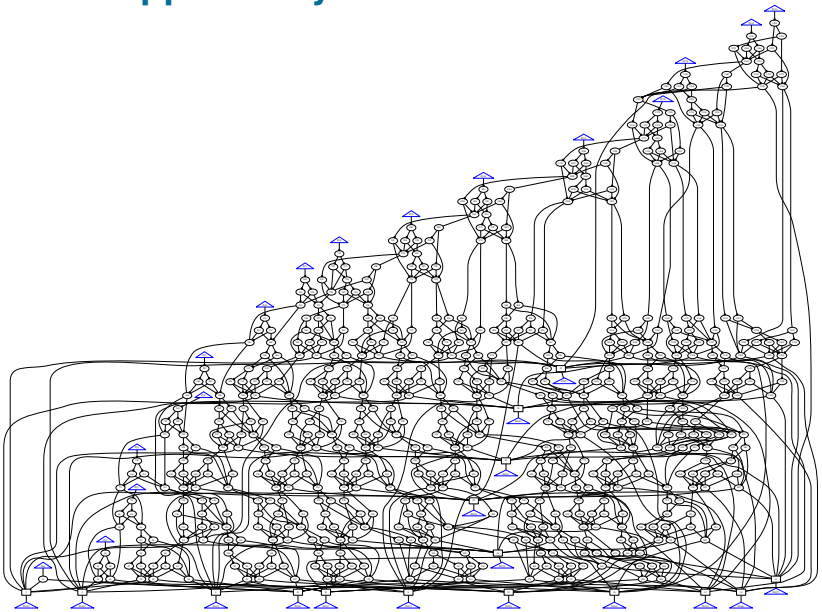
$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + s_0 - 2a_1 b_0 - a_0 b_0 \\
s_0 - a_0 b_0
\end{aligned}
$$

Verification

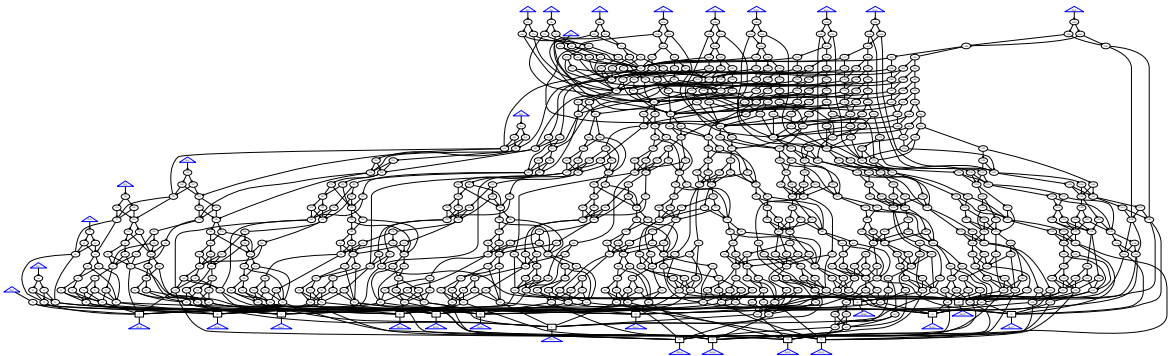
$$
\begin{aligned}
G(C) \cup B(C) = \{ & \\
& -s_3 + g_1 g_4, \\
& -s_2 - 2g_1 g_4 + g_4 + g_1, \\
& -g_4 + g_2 g_3, \\
& -s_1 - 2g_2 g_3 + g_3 + g_2, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_1 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
& -b_1^2 + b_1, \\
& -b_0^2 + b_0 \}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + s_0 - 2a_1 b_0 - a_0 b_0 \\
s_0 - a_0 b_0 \\
0
\end{aligned}
$$

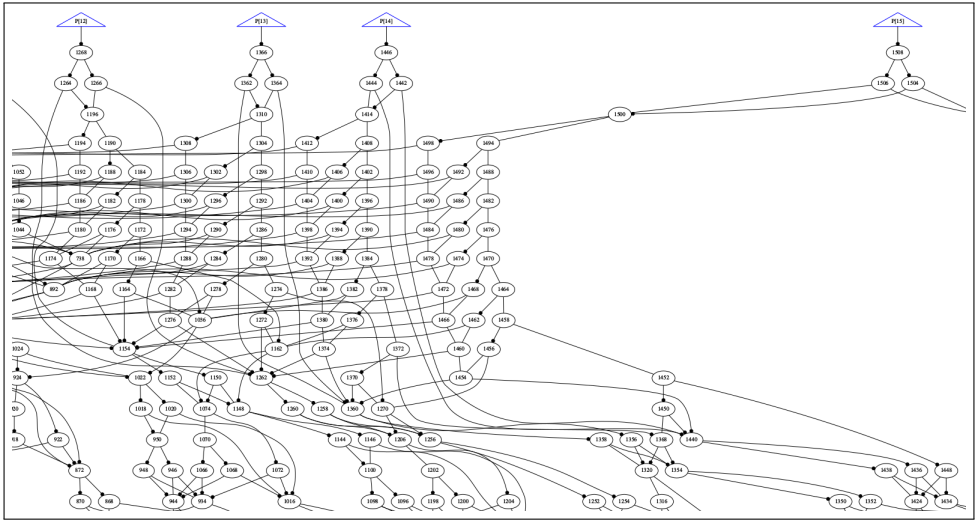
Multiplier – Ripple-Carry Adder



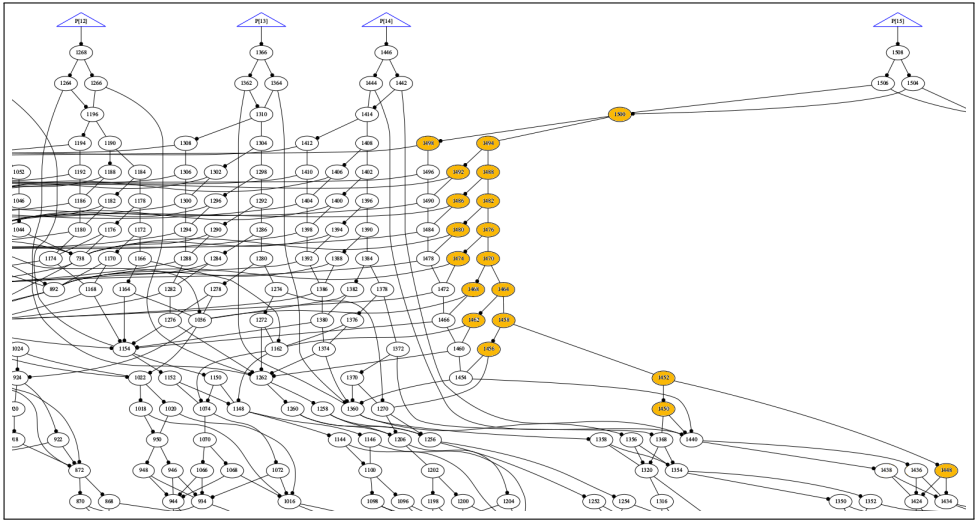
Multiplier – Carry-Lookahead Adder



Multiplier – Carry-Lookahead Adder

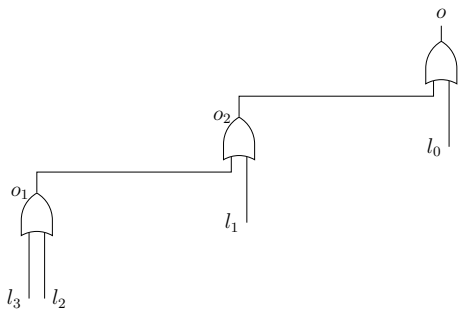


Multiplier – Carry-Lookahead Adder



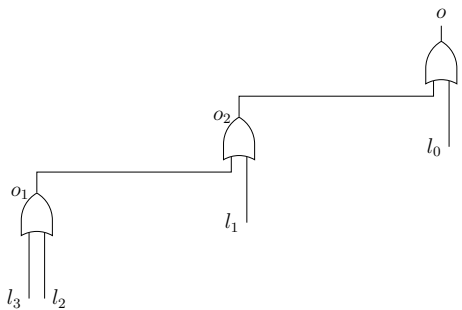
OR Gates

$$
\begin{aligned}
o = o_2 \vee x_0 \quad & -o + o_2 + l_0 - o_2 l_0, \\
o_2 = o_1 \vee l_1 \quad & -o_2 + o_1 + l_1 - o_1 l_1, \\
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$



OR Gates

$$
\begin{aligned}
o = o_2 \vee x_0 \quad & -o + o_2 + l_0 - o_2 l_0, \\
o_2 = o_1 \vee l_1 \quad & -o_2 + o_1 + l_1 - o_1 l_1, \\
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$

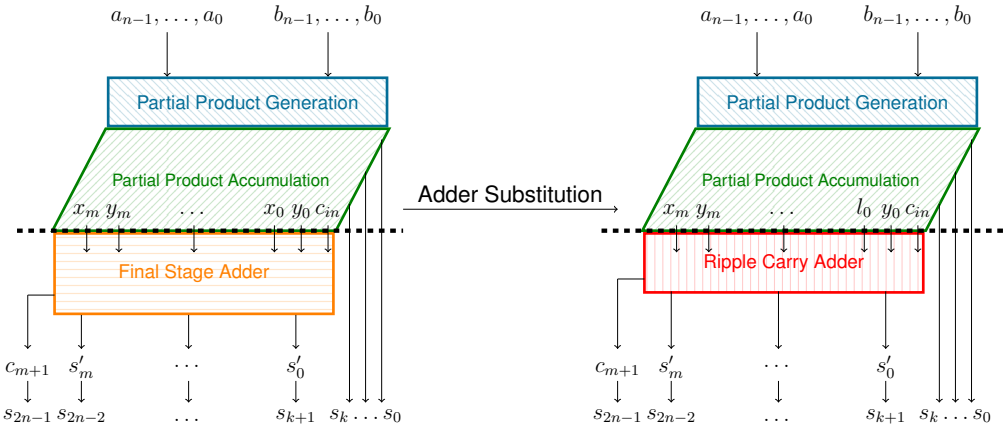


$$o = l_0 + l_1 - l_0 l_1 + l_2 - l_0 l_2 - l_1 l_2 + l_0 l_1 l_2 + l_3 - l_0 l_3 - l_1 l_3 + l_0 l_1 l_3 - l_2 l_3 + l_0 l_2 l_3 + l_1 l_2 l_3 - l_0 l_1 l_2 l_3$$

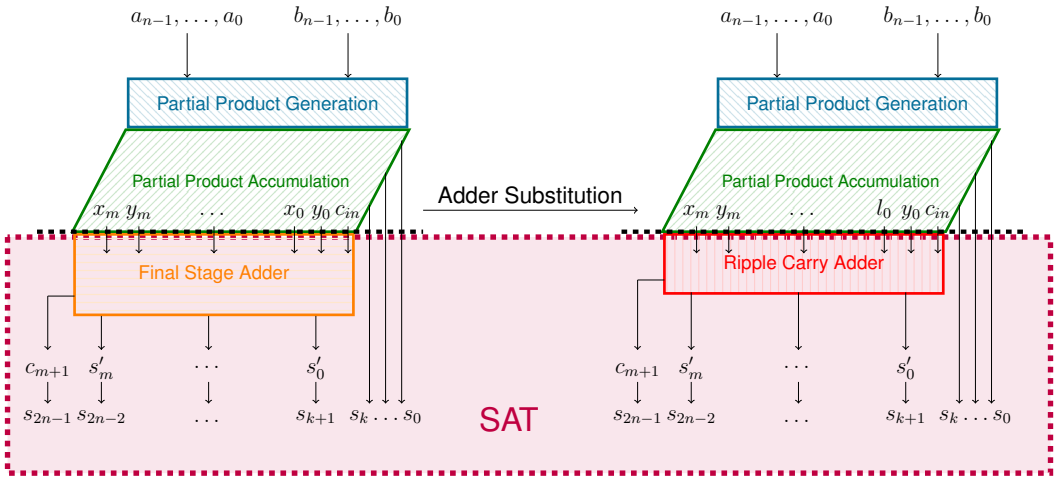
$15 = 2^4 - 1$ monomials

n OR Gates $\rightarrow 2^{n+1} - 1$ monomials

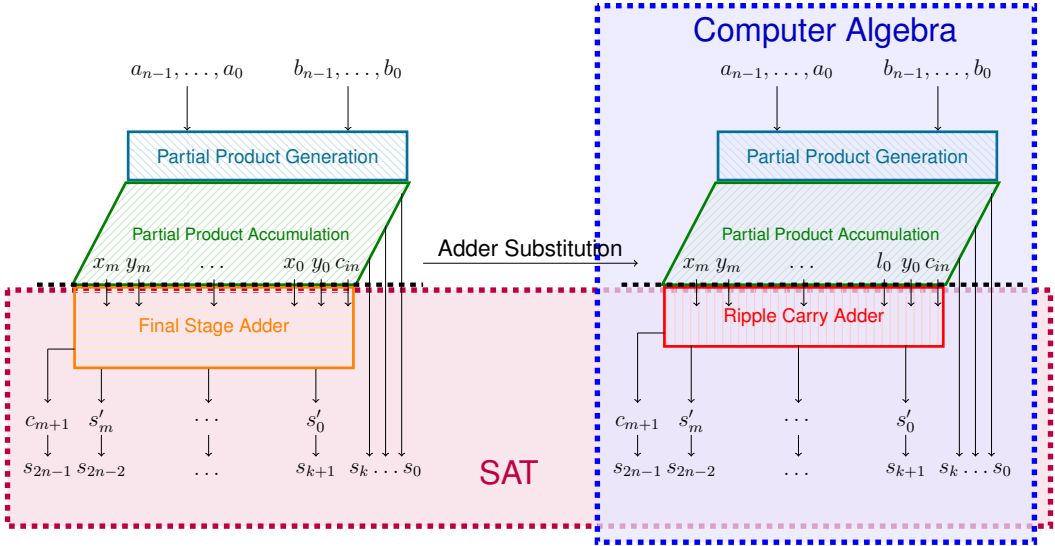
Explicit combination: SAT & Computer Algebra



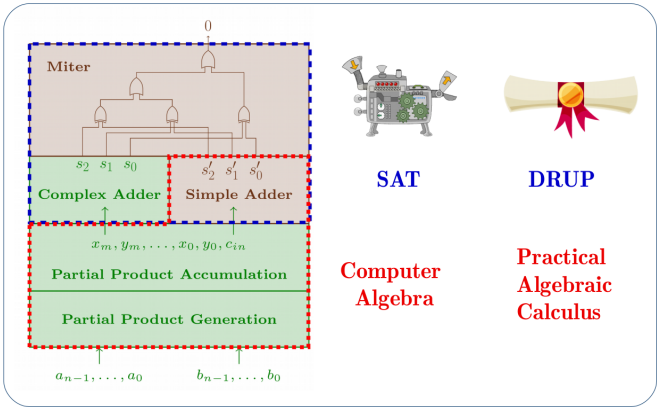
Explicit combination: SAT & Computer Algebra



Explicit combination: SAT & Computer Algebra



Problem: Proof Certificates

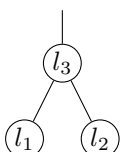


It is possible to simulate DRUP proofs in PAC, but it does not scale.

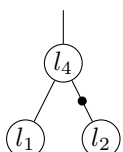
Dual Variables

[KaufmannBeameBiereNordström DATE'22]

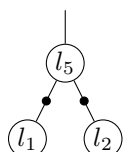
Provide a shorthand notation for inverters.



$$l_3 = l_1 \wedge l_2$$
$$-l_3 + l_1 l_2$$



$$l_4 = l_1 \wedge \neg l_2$$
$$-l_4 - l_1 l_2 + l_1$$



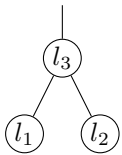
$$l_5 = \neg l_1 \wedge \neg l_2$$
$$-l_5 + l_1 l_2 - l_1 - l_2 + 1$$

Dual Variables

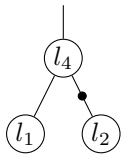
[KaufmannBeameBiereNordström DATE'22]

Provide a shorthand notation for inverters.

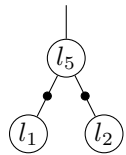
Dual variables.
Whenever two variables $l_i, f_i \in \{0, 1\}$ fulfill the relation $f_i = 1 - l_i$, we have $f_i = \mathrm{dual}(l_i)$.



$$l_3 = l_1 \wedge l_2$$
$$-l_3 + l_1 l_2$$
$$-l_3 + l_1 l_2$$



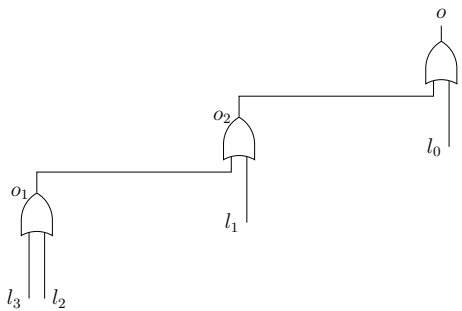
$$l_4 = l_1 \wedge \neg l_2$$
$$-l_4 - l_1 l_2 + l_1$$
$$-l_4 + l_1 f_2$$



$$l_5 = \neg l_1 \wedge \neg l_2$$
$$-l_5 + l_1 l_2 - l_1 - l_2 + 1$$
$$-l_5 + f_1 f_2$$

OR Gates

$$
\begin{aligned}
o = o_2 \vee x_0 \quad & -o + o_2 + l_0 - o_2 l_0, \\
o_2 = o_1 \vee l_1 \quad & -o_2 + o_1 + l_1 - o_1 l_1, \\
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$



OR Gates

$$
\begin{aligned}
o = o_2 \vee x_0 \quad & -o + o_2 + l_0 - o_2 l_0, \\
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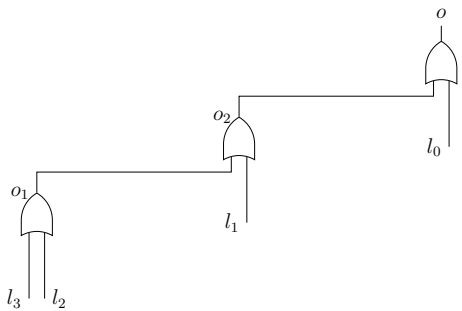
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$



$$o = l_0 + l_1 - l_0 l_1 + l_2 - l_0 l_2 - l_1 l_2 + l_0 l_1 l_2 + l_3 - l_0 l_3 - l_1 l_3 + l_0 l_1 l_3 - l_2 l_3 + l_0 l_2 l_3 + l_1 l_2 l_3 - l_0 l_1 l_2 l_3$$

$$o = 1 - f_0 f_1 f_2 f_3$$

Practical Difficulty

Key Method for polynomial inference: Gröbner basis algorithm

Relies on a reduction method based on a fixed variable order that will immediately eliminate one of each pair of dual variables by re-expressing it using its partner.

Practice: During verification we always reduce the specification by the dual constraint $-f_i - l_i + 1$ of a gate variable l_i before reducing by its gate constraint. This has the effect that all occurrences of f_i in the specification will be flipped to l_i before reducing l_i.

Problem: Compact representation is unfolded.

Practical Difficulty

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Problem: Compact representation is unfolded.

$\rightarrow$ We need dedicated heuristics to keep compact representation.

Calculate with Dual Variables

Proposition 1.
For all Boolean variables l_i and their dual representation $\mathrm{dual}(l_i) = f_i$ we have $l_i f_i = 0$.

"l_i and $\mathrm{dual}(l_i)$ cannot be 1 at the same time."

Proposition 2.
For all Boolean variables l_i and their dual representation $\mathrm{dual}(l_i) = f_i$ we have $l_i + f_i = 1$.

"l_i and $\mathrm{dual}(l_i)$ add up to 1."

Dual Mergeable

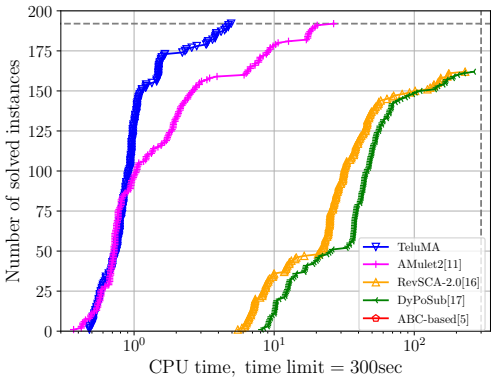
Example

Let $p = l_1 f_2 f_3 + l_1 f_2 l_3 + l_1 l_2 f_3 + f_1 f_2 + l_2 \in \mathbb{Z}[l_1, l_2, l_3, f_1, f_2, f_3]$. We write q_i to denote the polynomial q after iteration i and indicate the dual merges.

$$
\begin{aligned}
q_0 &= l_1 f_2 f_3 + l_1 f_2 l_3 + l_1 l_2 f_3 + f_1 f_2 + l_2 && r = 0 \\
q_1 &= l_1 l_2 f_3 + f_1 f_2 + \boxed{l_1 f_2} + l_2 && r = 0 \\
q_2 &= f_1 f_2 + l_1 f_2 + l_2 && r = l_1 l_2 f_3 \\
q_3 &= \boxed{f_2} + l_2 && r = l_1 l_2 f_3 \\
q_4 &= \boxed{1} && r = l_1 l_2 f_3 \\
q_5 &= 0 && r = l_1 l_2 f_3 + 1
\end{aligned}
$$

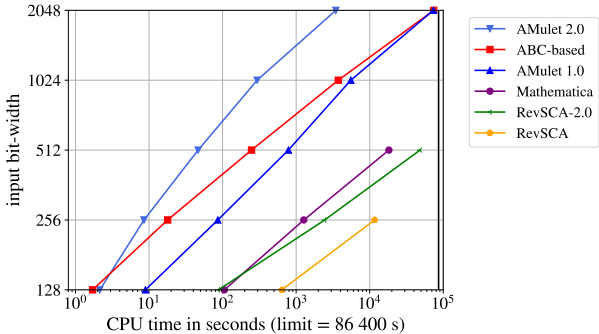
Evaluation - 64-bit Multipliers

Verification of 192 unsigned 64-bit multipliers



- **TeluMA** [KaufmannBeameBiereNordström DATE'22]
  variable elimination - incremental reduction - dual var.
- **AMulet2** [KaufmannBiere TACAS'21]
  variable elimination - incremental reduction - SAT
- **RevSCA-2.0** [MahzoonGroßeDrechsler DAC'19]
  variable elimination
- **DyPoSub** [MahzoonGroßeSchollDrechsler DATE'20]
  variable elimination - dynamic reduction

Evaluation - Large Multipliers



- Array ripple-carry multiplier
- Input bit-width n in $[128, 2048]$
- Time-limit: 86 400 sec (24h)

Conclusion

For multiplier verification using a . . .

. . . pure SAT encoding is  .

. . . pure algebraic encoding is  .

. . . combination of an algebraic and SAT encoding is  .

Conclusion

For multiplier verification using a . . .
. . . pure SAT encoding is  .
. . . pure algebraic encoding is  .
. . . combination of an algebraic and SAT encoding is  .

Future Work

- How to derive a minimal canonical encoding of polynomials using dual variables?
- How to generalize SAT concepts on polynomials?

COMBINING SAT AND COMPUTER ALGEBRA FOR MULTIPLIER VERIFICATION

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Dagstuhl Seminar
SAT Encodings and Beyond
Dagstuhl, Germany

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