SATISFIABLE ALGEBRAIC CIRCUIT VERIFICATION

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Bugs in hardware are expensive!

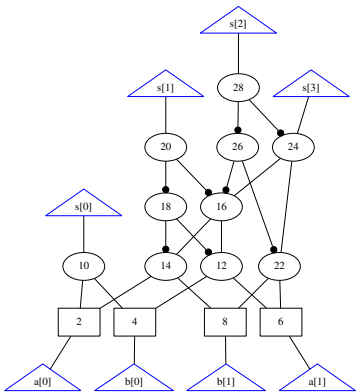
Circuit verification prevents issues like the famous Pentium FDIV bug.

Multiplier verification

Given: Gate-level integer multiplier for fixed bit-width.
Input format: AND-Inverter Graph

Question: For all possible $a_i, b_i \in \mathbb{B}$:

$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$



Formal Verification Techniques

Satisfiability Checking (SAT)

- SAT 2016 Competition
- Exponential run-time of solvers

Decision Diagrams

- First technique to detect Pentium bug
- Rely on manual decomposition

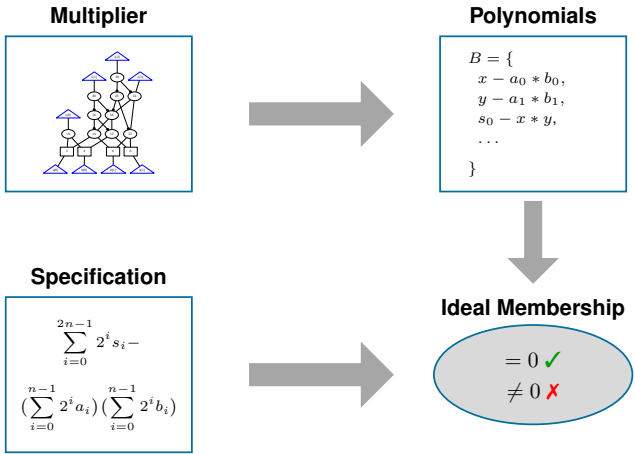
Theorem Proving

- Used in industry
- Requires manual effort
- Automated techniques rely on hierarchical information.

Algebraic Approach

- Great progress since 2015
- Polynomial encoding
- Works for non-trivial multiplier designs

Basic Idea of Algebraic Approach



From Circuits to Polynomials

Gate polynomials $G(C) \subset \mathbb{Z}[X]$.

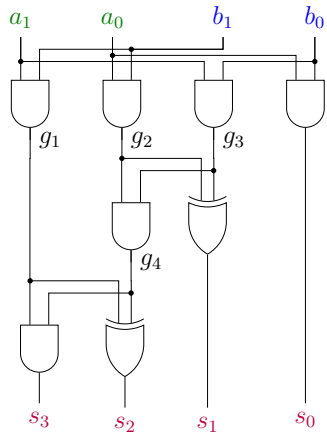
$$
\begin{array}{ll}
s_3 = g_1 \wedge g_4 & -s_3 + g_4 g_1, \\
s_2 = g_1 \oplus g_4 & -s_2 - 2g_4 g_1 + g_4 + g_1, \\
g_4 = g_2 \wedge g_3 & -g_4 + g_2 g_3, \\
s_1 = g_2 \oplus g_3 & -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 = a_1 \wedge b_1 & -g_1 + a_1 b_1, \\
g_2 = a_0 \wedge b_1 & -g_2 + a_0 b_1, \\
g_3 = a_1 \wedge b_0 & -g_3 + a_1 b_0, \\
s_0 = a_0 \wedge b_0 & -s_0 + a_0 b_0
\end{array}
$$

Boolean value constraints $B(C) \subset \mathbb{Z}[X]$.

$$
\begin{array}{ll}
a_1, a_0 \in \mathbb{B} & a_1(1-a_1),\ a_0(1-a_0), \\
b_1, b_0 \in \mathbb{B} & b_1(1-b_1),\ b_0(1-b_0)
\end{array}
$$

Specification $\mathcal{S}_n \in \mathbb{Z}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1 a_1 - 2b_1 a_0 - 2b_0 a_1 - b_0 a_0$$



Verification Technique

Verification Algorithm

Reduce specification $\sum_{i=0}^{2n-1} 2^i s_i - (\sum_{i=0}^{n-1} 2^i a_i)(\sum_{i=0}^{n-1} 2^i b_i)$ by elements of $G(C) \cup B(C)$

based on a fixed variable order until no further reduction is possible.

Then C is a multiplier iff the final remainder is zero.

Easy: Multipliers containing a ripple-carry adder

Hard: Multipliers containing a generate-and-propagate adder, e.g., carry-lookahead adder

Verification

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&\quad -s_3 + g_1 g_4, \\
&\quad -s_2 - 2g_1 g_4 + g_4 + g_1, \\
&\quad -g_4 + g_2 g_3, \\
&\quad -s_1 - 2g_2 g_3 + g_3 + g_2, \\
&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_1 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
&\quad -b_1^2 + b_1, \\
&\quad -b_0^2 + b_0\}
\end{aligned}
$$

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

Verification

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&-s_3 + g_1 g_4, \\
&-s_2 - 2g_1 g_4 + g_4 + g_1, \\
&-g_4 + g_2 g_3, \\
&-s_1 - 2g_2 g_3 + g_3 + g_2, \\
&-g_1 + a_1 b_1, \\
&-g_2 + a_0 b_1, \\
&-g_3 + a_1 b_0, \\
&-s_0 + a_0 b_0, \\
&-a_1^2 + a_1, \\
&-a_0^2 + a_0, \\
&-b_1^2 + b_1, \\
&-b_0^2 + b_0\}
\end{aligned}
$$

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

Verification

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\begin{aligned}
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&\quad -s_1 - 2g_2 g_3 + g_3 + g_2, \\
&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_1 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
&\quad -b_1^2 + b_1, \\
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\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0
\end{aligned}
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Verification

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&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_1 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
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4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
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Verification

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&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_1 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
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\end{aligned}
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8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
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4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
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2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
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2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
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\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1b_1 - 2a_1b_0 - 2a_0b_1 - a_0b_0 \\
8g_1g_4 + 4s_2 + 2s_1 + s_0 - 4a_1b_1 - 2a_1b_0 - 2a_0b_1 - a_0b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1b_1 - 2a_1b_0 - 2a_0b_1 - a_0b_0 \\
4g_2g_3 + 4g_1 + 2s_1 + s_0 - 4a_1b_1 - 2a_1b_0 - 2a_0b_1 - a_0b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1b_1 - 2a_1b_0 - 2a_0b_1 - a_0b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1b_0 - 2a_0b_1 - a_0b_0 \\
2g_3 + s_0 - 2a_1b_0 - a_0b_0 \\
s_0 - a_0b_0 \\
0
\end{aligned}
$$

Faulty multiplier

Gate polynomials $G(C)$.

$$
\begin{aligned}
s_3 &= g_1 \wedge g_4 && -s_3 + g_4 g_1, \\
s_2 &= g_1 \oplus g_4 && -s_2 - 2g_4 g_1 + g_4 + g_1, \\
g_4 &= g_2 \wedge g_3 && -g_4 + g_2 g_3, \\
s_1 &= g_2 \oplus g_3 && -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 &= a_1 \wedge b_1 && -g_1 + a_1 b_1, \\
g_2 &= a_0 \wedge b_1 && -g_2 + a_0 b_1, \\
g_3 &= a_1 \wedge b_0 && -g_3 + a_1 b_0, \\
s_0 &= a_0 \wedge b_0 && -s_0 + a_0 b_0
\end{aligned}
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Boolean value constraints $B(C)$.

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\begin{aligned}
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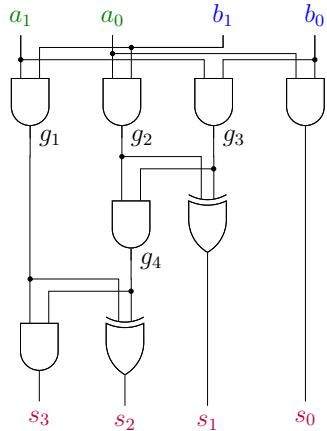

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g_4 &= g_2 \wedge g_3 && -g_4 + g_2 g_3, \\
s_1 &= g_2 \oplus g_3 && -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 &= a_1 \wedge b_1 && -g_1 + a_1 b_1, \\
g_2 &= a_0 \wedge b_1 && -g_2 + a_0 b_1, \\
g_3 &= a_1 \wedge b_0 && -g_3 + a_1 b_0, \\
s_0 &= a_0 \wedge b_0 && -s_0 + a_0 b_0
\end{aligned}
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a_1, a_0 &\in \mathbb{B} && a_1(1-a_1),\ a_0(1-a_0), \\
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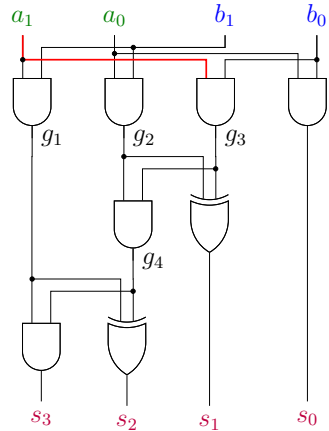

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g_4 &= g_2 \wedge g_3 && -g_4 + g_2 g_3, \\
s_1 &= g_2 \oplus g_3 && -s_1 - 2g_2 g_3 + g_2 + g_3, \\
g_1 &= a_1 \wedge b_1 && -g_1 + a_1 b_1, \\
g_2 &= a_0 \wedge b_1 && -g_2 + a_0 b_1, \\
g_3 &= a_1 \wedge b_0 && -g_3 + a_0 b_0, \\
s_0 &= a_0 \wedge b_0 && -s_0 + a_0 b_0
\end{aligned}
$$

Boolean value constraints $B(C)$.

$$
\begin{aligned}
a_1, a_0 &\in \mathbb{B} && a_1(1-a_1),\ a_0(1-a_0), \\
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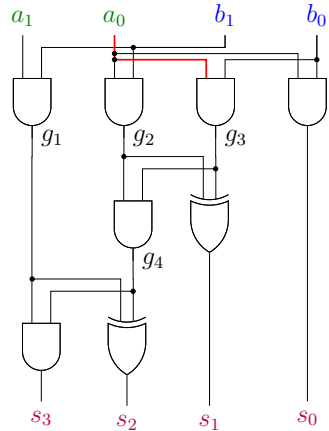
\end{aligned}
$$



Verification of a faulty multiplier

$$
\begin{aligned}
G(C) \cup B(C) = \{ & \\
& -s_3 + g_1 g_4, \\
& -s_2 - 2g_1 g_4 + g_4 + g_1, \\
& -g_4 + g_2 g_3, \\
& -s_1 - 2g_2 g_3 + g_3 + g_2, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_0 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
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\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
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4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + s_0 - 2a_1 b_0 - a_0 b_0
\end{aligned}
$$

Verification of a faulty multiplier

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
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&\quad -g_4 + g_2 g_3, \\
&\quad -s_1 - 2g_2 g_3 + g_3 + g_2, \\
&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_0 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
&\quad -b_1^2 + b_1, \\
&\quad -b_0^2 + b_0 \}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + s_0 - 2a_1 b_0 - a_0 b_0 \\
s_0 - 2a_1 b_0 + a_0 b_0
\end{aligned}
$$

Verification of a faulty multiplier

$$
\begin{aligned}
&G(C) \cup B(C) = \{ \\
&\quad -s_3 + g_1 g_4, \\
&\quad -s_2 - 2g_1 g_4 + g_4 + g_1, \\
&\quad -g_4 + g_2 g_3, \\
&\quad -s_1 - 2g_2 g_3 + g_3 + g_2, \\
&\quad -g_1 + a_1 b_1, \\
&\quad -g_2 + a_0 b_1, \\
&\quad -g_3 + a_0 b_0, \\
&\quad -s_0 + a_0 b_0, \\
&\quad -a_1^2 + a_1, \\
&\quad -a_0^2 + a_0, \\
&\quad -b_1^2 + b_1, \\
&\quad -b_0^2 + b_0 \}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + s_0 - 2a_1 b_0 - a_0 b_0 \\
s_0 - 2a_1 b_0 + a_0 b_0
\end{aligned}
$$

Verification of a faulty multiplier

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& -s_2 - 2g_1 g_4 + g_4 + g_1, \\
& -g_4 + g_2 g_3, \\
& -s_1 - 2g_2 g_3 + g_3 + g_2, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_0 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, \\
& -a_0^2 + a_0, \\
& -b_1^2 + b_1, \\
& -b_0^2 + b_0 \}
\end{aligned}
$$

$$
\begin{aligned}
8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
8g_1 g_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_4 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_2 g_3 + 4g_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
4g_1 + 2g_3 + 2g_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + 2g_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0 \\
2g_3 + s_0 - 2a_1 b_0 - a_0 b_0 \\
s_0 - 2a_1 b_0 + a_0 b_0 \\
-2a_1 b_0 + 2a_0 b_0
\end{aligned}
$$

Counter Example

Remainder: $-2a_1b_0 + 2a_0b_0 \neq 0$

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$a_1 = 0, \quad a_0 = 1$
$b_1 = 0, \quad b_0 = 1$

$01 \cdot 01 = 0001$

Counter Example

Remainder: $-2a_1b_0 + 2a_0b_0 \neq 0$

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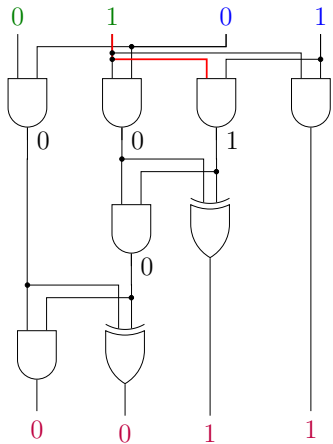
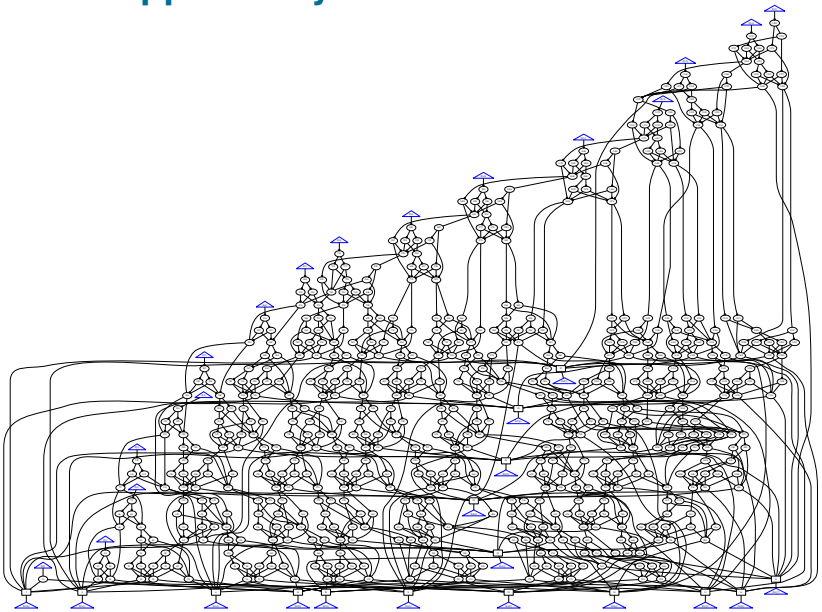
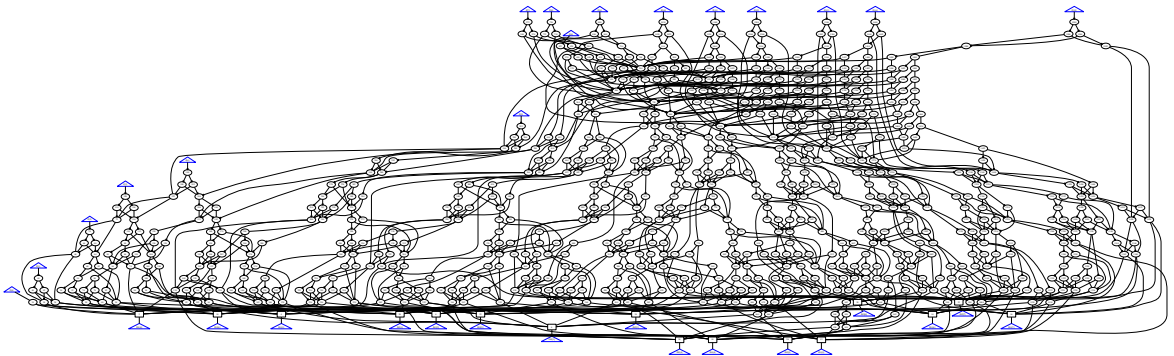


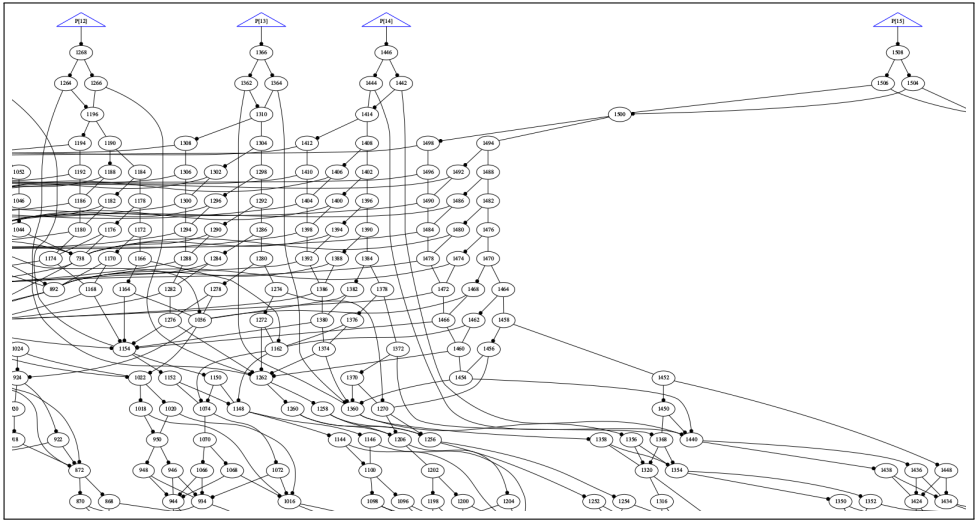
Multiplier – Ripple-Carry Adder



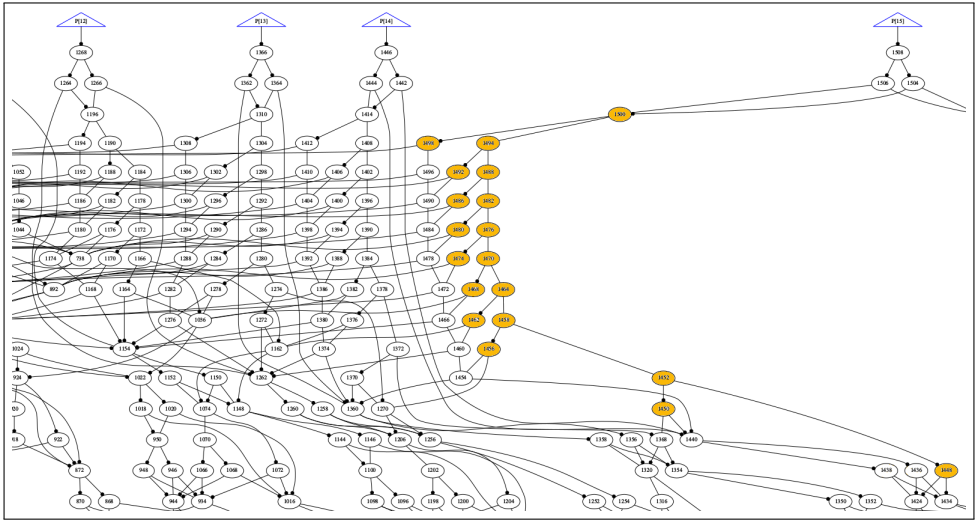
Multiplier – Carry-Lookahead Adder



Multiplier – Carry-Lookahead Adder

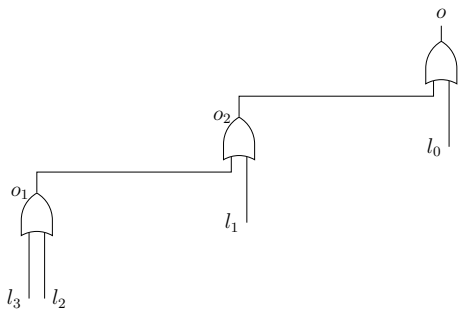


Multiplier – Carry-Lookahead Adder



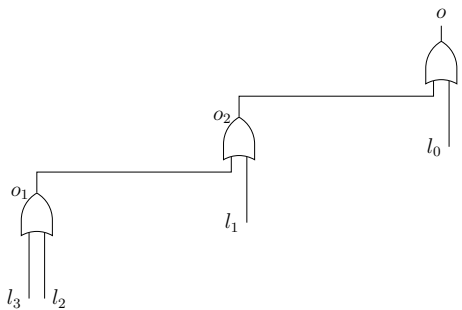
OR Gates

$$
\begin{aligned}
o = o_2 \vee x_0 \quad & -o + o_2 + l_0 - o_2 l_0, \\
o_2 = o_1 \vee l_1 \quad & -o_2 + o_1 + l_1 - o_1 l_1, \\
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$



OR Gates

$$
\begin{aligned}
o = o_2 \vee x_0 \quad & -o + o_2 + l_0 - o_2 l_0, \\
o_2 = o_1 \vee l_1 \quad & -o_2 + o_1 + l_1 - o_1 l_1, \\
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$



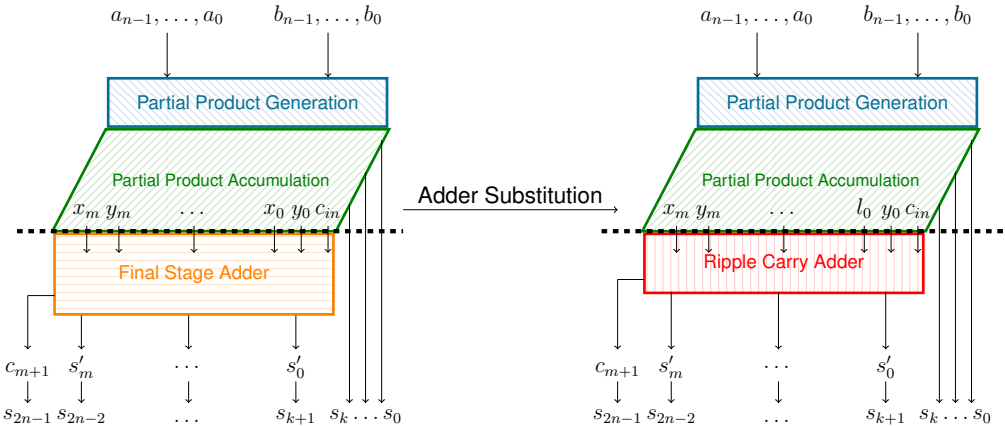
$$o = l_0 + l_1 - l_0 l_1 + l_2 - l_0 l_2 - l_1 l_2 + l_0 l_1 l_2 + l_3 - l_0 l_3 - l_1 l_3 + l_0 l_1 l_3 - l_2 l_3 + l_0 l_2 l_3 + l_1 l_2 l_3 - l_0 l_1 l_2 l_3$$

$15 = 2^4 - 1$ monomials

n OR Gates $\rightarrow 2^{n+1} - 1$ monomials

Previous Approach: SAT & Computer Algebra

[Kaufmann et al., 2019]



Previous Approach: SAT & Computer Algebra

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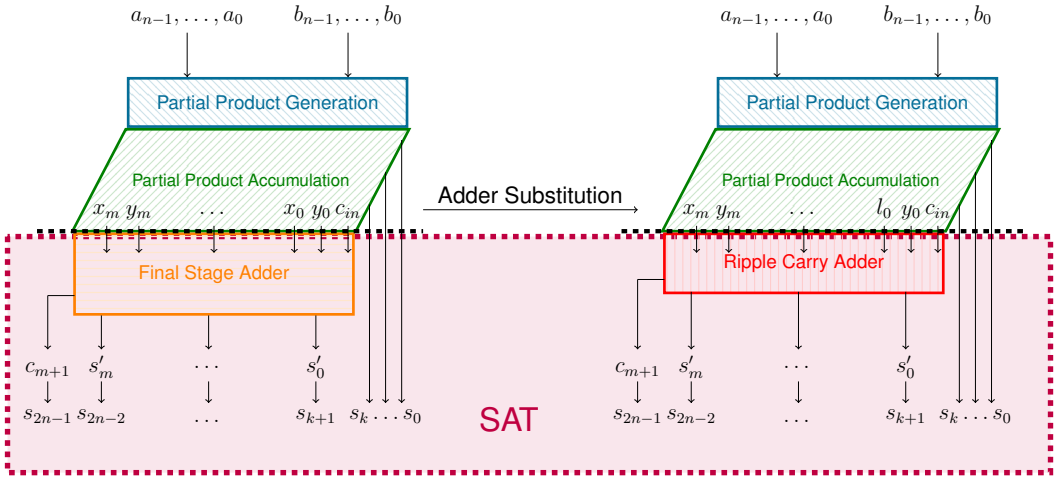

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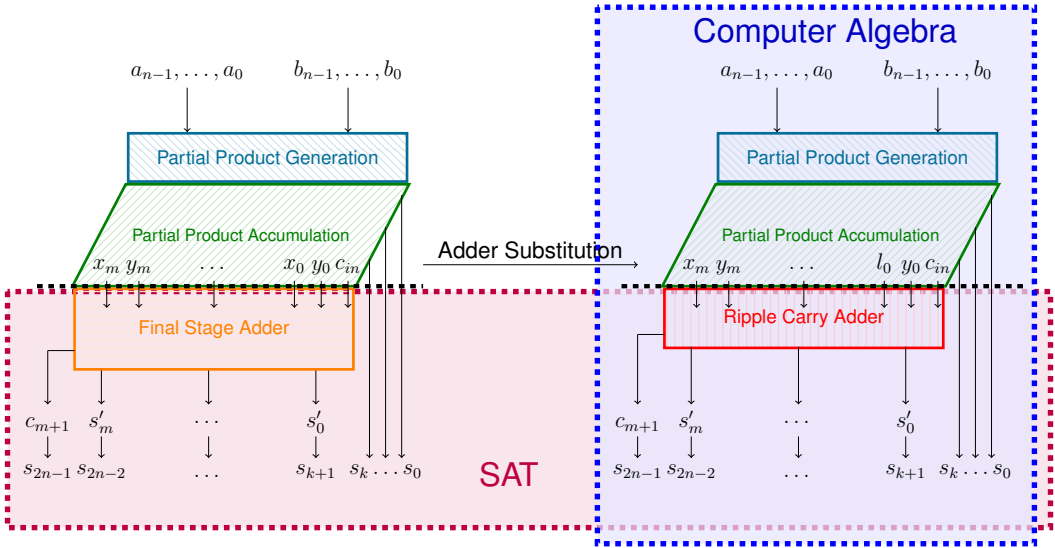
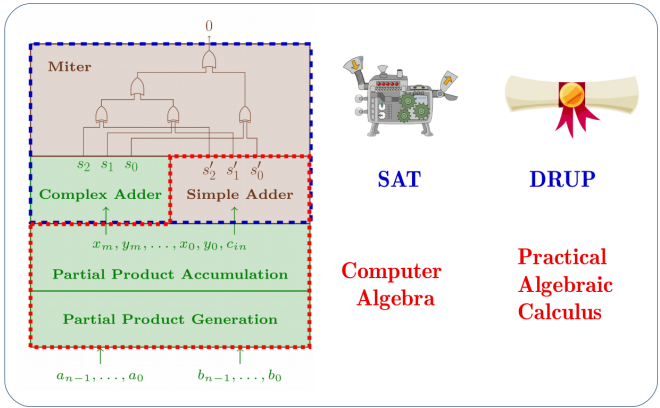


Problem: Proof Certificates



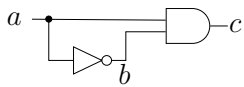
It is possible to simulate DRUP proofs in PAC, but it does not scale [Kaufmann et al., 2020a].

Challenges

How to combine DRUP and PAC into a single proof?

Practical Algebraic Calculus

[Ritirc et al., 2018]



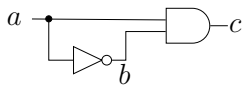

```
P =   -b+1-a,        b = ¬a
      -c+a*b,        c = a ∧ b = a ∧ ¬a
      -a^2+a         a = ⊥ ∨ a = ⊤
Spec = c             c = ⊥
```

```
* : -b+1-a,          a,          -a*b+a-a^2;
* : -a^2+a,          -1,         a^2-a;
+ : -a*b+a-a^2,      a^2-a,      -a*b;
+ : -a*b,            -c+a*b,     -c;
* : -c,              -1,         c;
```

$\forall\, p, q \in I : p + q \in I$ and $\forall\, p \in \mathbb{Z}[X]\ \forall\, q \in I : pq \in I$

Practical Algebraic Calculus

[Ritirc et al., 2018]




```
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Spec = c           c = ⊥
```
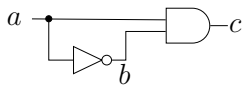

```
* : -b+1-a,        a,          -a*b+a-a^2;
* : -a^2+a,        -1,         a^2-a;
+ : -a*b+a-a^2,    a^2-a,      -a*b;
+ : -a*b,          -c+a*b,     -c;
* : -c,            -1,         c;
```

Can be checked by our older proof checker PACTRIM.

1. Boolean Variables

Handle Boolean-value constraints implicitly to reduce number of proof steps.

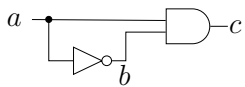



```
P =   -b+1-a,
      -c+a*b,
      -a^2+a
Spec = c
```

```
* : -b+1-a,      a,        -a*b;
+ : -a*b,        -c+a*b,   -c;
* : -c,          -1,       c;
```

2. Indices

Introduce indices to reduce proof size.

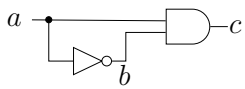



```
P =   1 -b+1-a;
      2 -c+a*b;
Spec = c
```

```
3   * 1,   a,   -a*b;
4   + 3,   2,   -c;
5   * 4,  -1,   c;
```

3. Deletion Rule

Introduce a deletion rule to reduce the memory usage of the proof checker.

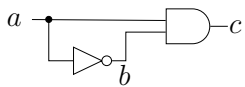



```
P =   1 -b+1-a;
      2 -c+a*b;
Spec = c
```

```
3   * 1,   a,   -a*b;
1   d;
4   + 3,   2,   -c;
2   d;
3   d;
5   * 4,  -1,   c;
```

4. Linear Combination Rule

Introduce a linear combination rule to replace explicit addition and multiplication rules.




```
P =   1 -b+1-a;
      2 -c+a*b;
Spec = c
```

```
3   % (-a)*1 + (-1)*2,  c;
```

5. Extension Rule

The extension rule allows to add model preserving polynomials to the constraint set.

$$\frac{\overline{x} \vee \overline{y} \qquad y \vee z}{\overline{x} \vee z}$$

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$$\frac{xy \qquad (1-y)(1-z)}{x(1-z)}$$

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$$\frac{xy \qquad (1-y)(1-z)}{x(1-z)}$$

```
P =   1 x*y;
      2 y*z-y-z+1;
Spec =  -x*z+x
```

```
3  =  f, -z+1;
4  *  3,  y-1, -f*y+f-y*z+y+z-1;
5  +  2,    4, -f*y+f;
```

Ext(i, v, p) $(X, P) \Rightarrow (X \cup \{v\}, P(i \mapsto -v + p))$
provided that $P(i) = \bot$ and $v \notin X$ and $p \in \mathbb{Z}[X]/\langle B(X)\rangle$,
and $p^2 - p \equiv 0 \mod \langle B(X)\rangle$.

5. Extension Rule

The extension rule allows to add model preserving polynomials to the constraint set.

$$\frac{xy \qquad (1-y)(1-z)}{x(1-z)}$$

```
P =   1 x*y;
      2 y*z-y-z+1;
Spec =  -x*z+x
```

```
 3  =  f, -z+1;
 4  *  3,  y-1, -f*y+f-y*z+y+z-1;
 5  +  2,    4, -f*y+f;
 6  *  1,    f, f*x*y;
 7  *  5,    x, -f*x*y+f*x;
 8  +  6,    7, f*x;
 9  *  3,    x, -f*x-x*z+x;
10  +  8,    9, -x*z+x;
```

PACHECK & PASTÈQUE

[Kaufmann et al., 2020b]

PACHECK C++ implementation

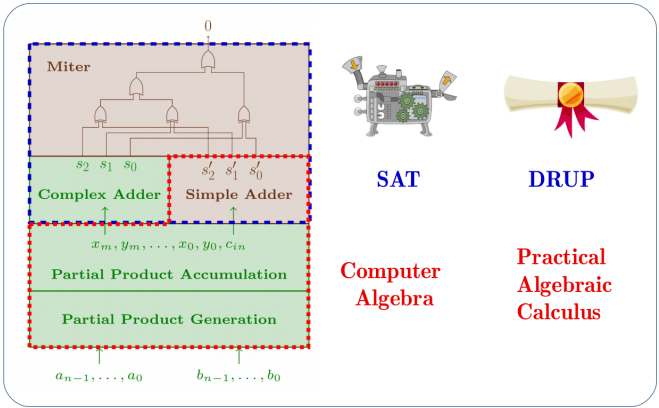
PASTÈQUE Isabelle/HOL

- relying on Isabelle's Refinement Framework
- abstract specification on ideals: specification in ideal
- final step: executable checker

Challenges

How to generalize **EXT** in a meaningful way?

Problem: Proof Certificates

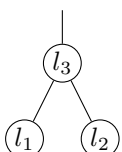


It is possible to simulate DRUP proofs in PAC, but it does not scale [Kaufmann et al., 2020a].

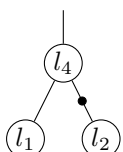
Dual Variables

[Kaufmann et al., 2022]

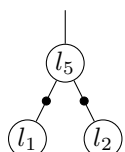
Provide a shorthand notation for inverters.



$$l_3 = l_1 \wedge l_2$$
$$-l_3 + l_1 l_2$$



$$l_4 = l_1 \wedge \neg l_2$$
$$-l_4 - l_1 l_2 + l_1$$



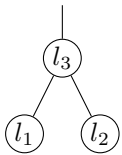
$$l_5 = \neg l_1 \wedge \neg l_2$$
$$-l_5 + l_1 l_2 - l_1 - l_2 + 1$$

Dual Variables

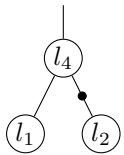
[Kaufmann et al., 2022]

Provide a shorthand notation for inverters.

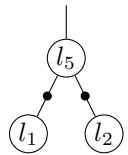
Dual variables.
Whenever two variables $l_i, f_i \in \{0, 1\}$ fulfill the relation $f_i = 1 - l_i$, we have $f_i = \mathrm{dual}(l_i)$.



$$l_3 = l_1 \wedge l_2$$
$$-l_3 + l_1 l_2$$
$$-l_3 + l_1 l_2$$



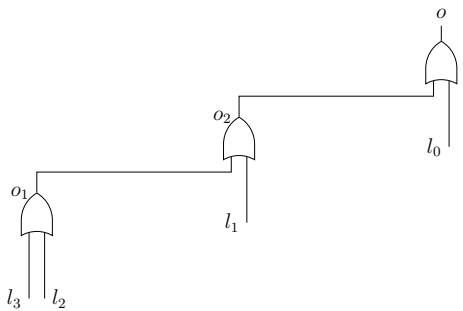
$$l_4 = l_1 \wedge \neg l_2$$
$$-l_4 - l_1 l_2 + l_1$$
$$-l_4 + l_1 f_2$$



$$l_5 = \neg l_1 \wedge \neg l_2$$
$$-l_5 + l_1 l_2 - l_1 - l_2 + 1$$
$$-l_5 + f_1 f_2$$

OR Gates

$$
\begin{aligned}
o = o_2 \vee x_0 \quad & -o + o_2 + l_0 - o_2 l_0, \\
o_2 = o_1 \vee l_1 \quad & -o_2 + o_1 + l_1 - o_1 l_1, \\
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$



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$$
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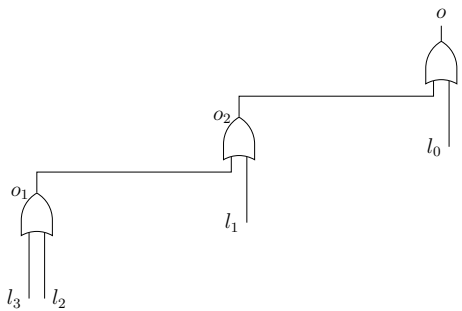
o_1 = l_3 \vee l_2 \quad & -o_1 + l_3 + l_2 - l_3 l_2
\end{aligned}
$$



$$o = l_0 + l_1 - l_0 l_1 + l_2 - l_0 l_2 - l_1 l_2 + l_0 l_1 l_2 + l_3 - l_0 l_3 - l_1 l_3 + l_0 l_1 l_3 - l_2 l_3 + l_0 l_2 l_3 + l_1 l_2 l_3 - l_0 l_1 l_2 l_3$$

$$o = 1 - f_0 f_1 f_2 f_3$$

Practical Difficulty

Key Method for polynomial inference: Gröbner basis algorithm

Relies on a reduction method based on a fixed variable order that will immediately eliminate one of each pair of dual variables by re-expressing it using its partner.

Practice: During verification we always reduce the specification by the dual constraint $-f_i - l_i + 1$ of a gate variable l_i before reducing by its gate constraint. This has the effect that all occurrences of f_i in the specification will be flipped to l_i before reducing l_i.

Problem: Compact representation is unfolded.

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Problem: Compact representation is unfolded.

$\rightarrow$ We need dedicated preprocessing techniques to keep compact representation.

Challenges

How to integrate dual variables into polynomial reduction?

Calculate with Dual Variables

Proposition 1.
For all Boolean variables l_i and their dual representation $\mathrm{dual}(l_i) = f_i$ we have $l_i f_i = 0$.

"l_i and $\mathrm{dual}(l_i)$ cannot be 1 at the same time."

Proposition 2.
For all Boolean variables l_i and their dual representation $\mathrm{dual}(l_i) = f_i$ we have $l_i + f_i = 1$.

"l_i and $\mathrm{dual}(l_i)$ add up to 1."

Dual Mergeable

We call two monomials m_1 and m_2 **dual mergeable** iff $m_1 = cf_i\tau$ and $m_2 = cl_i\tau$ for c a constant, τ a term, and some index i. We call the monomial $\mathrm{dmerge}(m_1, m_2) = c\tau$ their **dual merge**.

Algorithm: Merging monomials(p)

Input : Polynomial p
Output: Simplified polynomial r

$q \leftarrow$ sort-degree-lex(p); $r \leftarrow 0$;
while $q \neq 0$ **do**
  $q_l \leftarrow \mathrm{lm}(q)$; $t \leftarrow \mathrm{tail}(q)$; simplify $\leftarrow \bot$;
  while $t \neq 0$ and $\deg(q_l) = \deg(\mathrm{lt}(t))$ and $\neg$ simplify **do**
    $q_t \leftarrow \mathrm{lt}(t)$;
    if q_l and q_t are dual mergeable **then**
      $q \leftarrow q - q_l - q_t + \mathrm{dmerge}(q_l, q_t)$;
      simplify $\leftarrow \top$;
    else $t \leftarrow t - q_t$;
  if $\neg$ simplify **then** $r \leftarrow r + q_l$, $q \leftarrow q - q_l$;
return sort-lex(r);

Dual Mergeable

Example

Let $p = l_1 f_2 f_3 + l_1 f_2 l_3 + l_1 l_2 f_3 + f_1 f_2 + l_2 \in \mathbb{Z}[l_1, l_2, l_3, f_1, f_2, f_3]$. We write q_i to denote the polynomial q after iteration i and indicate the dual merges.

$$
\begin{aligned}
q_0 &= l_1 f_2 f_3 + l_1 f_2 l_3 + l_1 l_2 f_3 + f_1 f_2 + l_2 && r = 0 \\
q_1 &= l_1 l_2 f_3 + f_1 f_2 + \boxed{l_1 f_2} + l_2 && r = 0 \\
q_2 &= f_1 f_2 + l_1 f_2 + l_2 && r = l_1 l_2 f_3 \\
q_3 &= \boxed{f_2} + l_2 && r = l_1 l_2 f_3 \\
q_4 &= \boxed{1} && r = l_1 l_2 f_3 \\
q_5 &= 0 && r = l_1 l_2 f_3 + 1
\end{aligned}
$$

Challenges

How to derive the most efficient representation of polynomials using dual variables?

How to apply other SAT concepts on polynomials?

Carry Rewriting

Goal: Rewrite encoding of carry look-ahead unit into a ripple-carry unit, which can easily be verified using computer algebra.

Algorithm: Carry-Rewriting

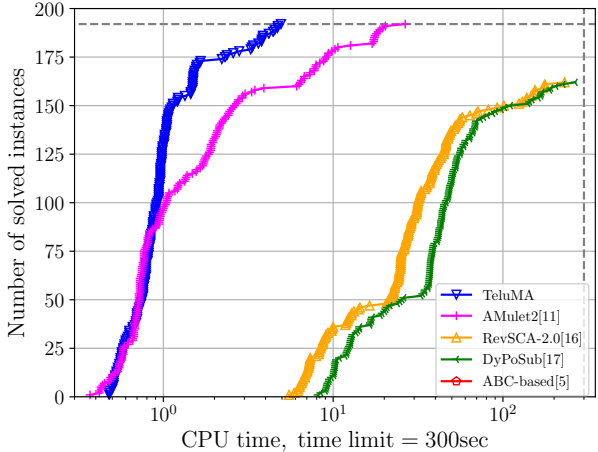
Input : Circuit C in AIG format

Output: Carry-rewritten Gröbner basis of C

1 $F \leftarrow$ Mark-final-stage-adder(C);
2 $G \leftarrow$ Dual-Polynomial-Encoding(F);
3 $H \leftarrow$ Polynomial-Encoding($C \setminus F$);
4 $G \leftarrow$ Eliminate-Pure-Positive-Variables(G);
5 $G \leftarrow$ Tail-Substitution(G);
6 $G \leftarrow$ Carry-Unfolding(G);
7 **return** $G \cup H$

Evaluation - Multiplier Verification

Verification of 192 unsigned 64-bit multipliers



Evaluation - Proof Certificates

architecture	n	[Kaufmann et al., 2019]			[Kaufmann et al., 2020a]		[Kaufmann et al., 2022]	
		DRUP	PAC	total (s)	PAC	total (s)	PAC	total (s)
		# rules	# rules		# rules		# rules	
sp-ar-cl	32	14 927	33 834	1	1 597 897	164	60 336	0
sp-bd-ks	32	17 528	34 958	1	817 956	28	54 116	0
sp-dt-lf	32	3 138	33 451	1	321 720	5	47 835	0
bp-ct-bk	32	2 276	27 312	1	217 128	3	36 356	0
bp-wt-cl	32	46 502	30 561	2	5 536 176	3 375	114 665	2
sp-ar-cl	64	65 317	139 338	8	-	TO	289 632	4
sp-bd-ks	64	44 921	142 138	6	1 440 943	74	214 378	3
sp-dt-lf	64	28 772	138 539	6	816 572	19	192 805	2
bp-ct-bk	64	19 891	105 579	5	459 262	15	136 703	2
bp-wt-cl	64	42 199	118 573	19	-	TO	774 044	24

All benchmarks are generated by the Arithmetic Model Generator [Homma et al., 2006]. TO = 3600 sec

Challenges

Proofs

- How to generalize **EXT** in a meaningful way?
- How to combine DRUP and PAC into a single proof?

Algebraic reasoning

- How to integrate dual variables into polynomial reduction?
- How to derive the most efficient representation of polynomials using dual variables?
- How to apply other SAT concepts on polynomials?

SATISFIABLE ALGEBRAIC CIRCUIT VERIFICATION

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