

TUTORIAL ON ALGEBRAIC VERIFICATION OF ARITHMETIC CIRCUITS: GRÖBNER BASES IN PRACTICE

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TU Wien, Austria

Invited Tutorial

18th Intl. Conf. on Verified Software: Theories, Tools and Experiments

September 15, 2026



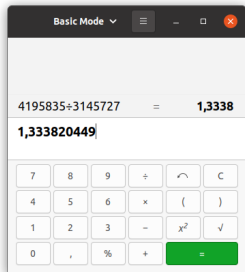
Informatics



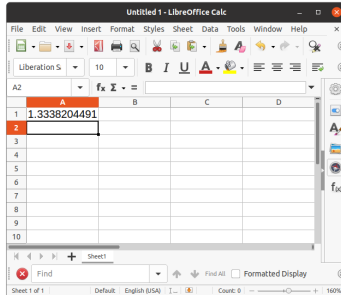
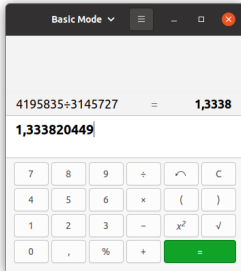
Austrian
Science Fund

$$4\,195\,835 \div 3\,145\,727 = ?$$

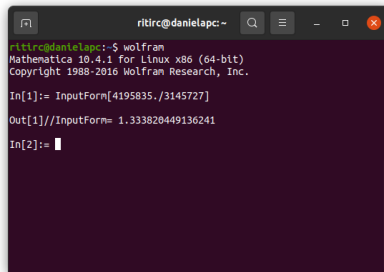
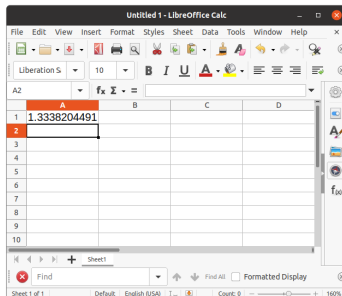
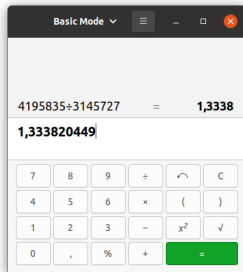
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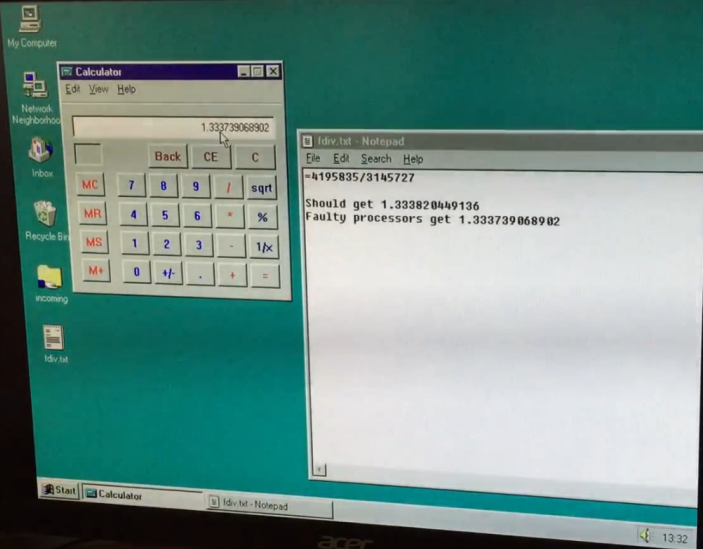


$$4\,195\,835 \div 3\,145\,727 = ?$$



$$4\,195\,835 \div 3\,145\,727 = ?$$





Intel Pentium FDIV-Bug 1994



Quelle: <http://neology.com.au/portfolios/a80502-90-sx923/>

- ⚙️ Affected floating point unit (FPU) in early Intel processors.
- ⚠️ Processor might return incorrect result for division.
- 💰 Cost in 1994: 500 million dollars.

Even more than 30 years later verification of arithmetic circuits is considered to be hard. Correctness proofs are not fully automated yet.

Challenge: Integer multipliers

Integer Multiplication

$$\begin{array}{r} 11 \\ \times 11 \\ \hline \end{array}$$

$$3 \cdot 3 = 9$$

Integer Multiplication

$$\begin{array}{r} 11 \cdot 11 \\ \hline 11 \end{array}$$

$$3 \cdot 3 = 9$$

Integer Multiplication

$$\begin{array}{r} 11 \cdot 11 \\ \hline 11 \\ 11 \\ \hline \end{array}$$

$$3 \cdot 3 = 9$$

Integer Multiplication

$$\begin{array}{r} 11 \cdot 11 \\ \hline 11 \\ 110 \\ \hline 121 \end{array}$$

$$3 \cdot 3 = 9$$

Integer Multiplication

$$\begin{array}{r} 11 \cdot 11 \\ \hline 11 \\ 110 \\ \hline 011 \end{array}$$

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Integer Multiplication

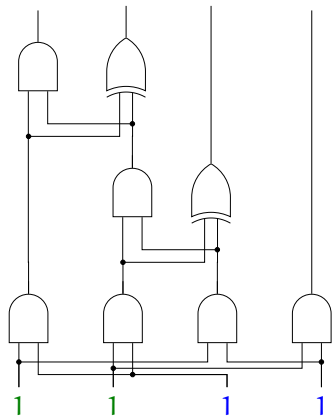
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Integer Multiplication

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Integer Multiplication

AND-Gate



$$f \wedge g = y$$

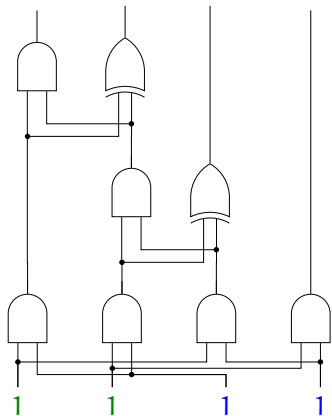
f	g	y
0	0	0
0	1	0
1	0	0
1	1	1

XOR-Gate



$$f \oplus g = y$$

f	g	y
0	0	0
0	1	1
1	0	1
1	1	0



Integer Multiplication

AND-Gate



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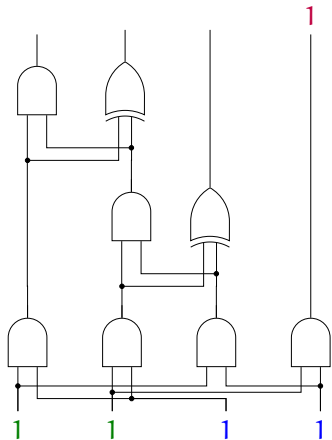
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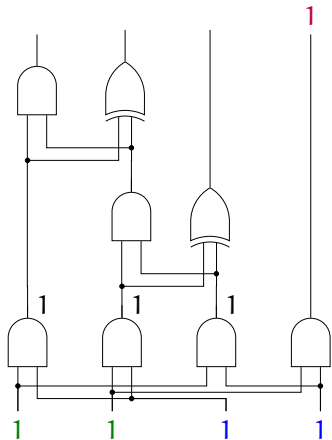
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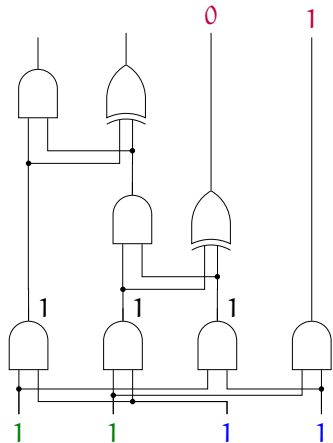
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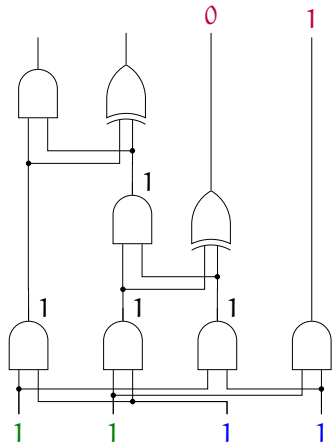
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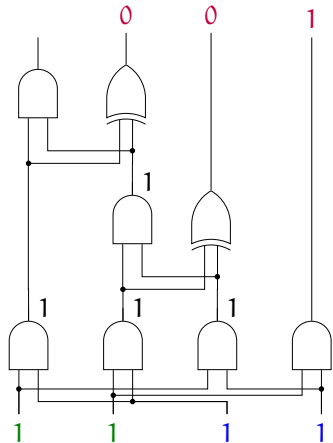
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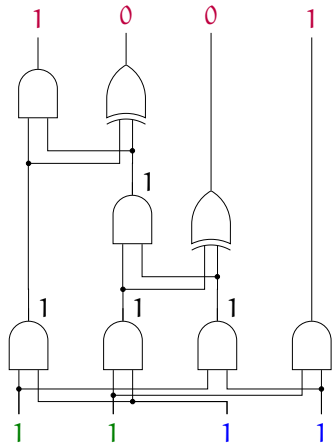
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XOR-Gate



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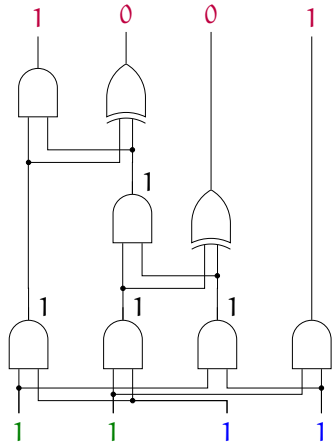
f	g	y
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1	0	1
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Is the circuit correct?

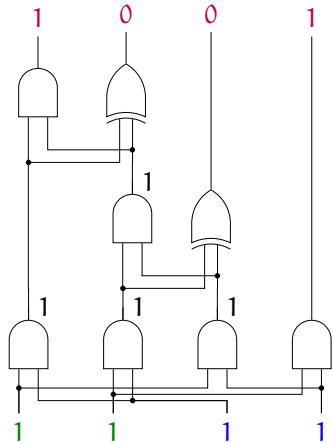
Multiplier circuit

$$\begin{array}{r} 11 \cdot 11 \\ \hline 11 \\ 110 \\ \hline 1001 \end{array}$$



Multiplier circuit

Circuit seems correct!



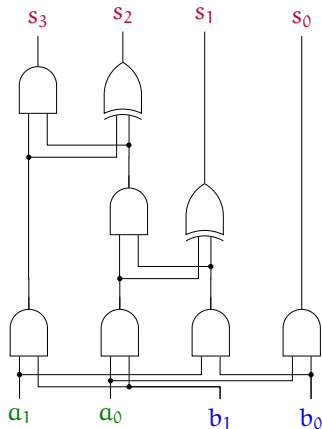
Is the circuit really correct?

Multiplier Circuits

Given: Gate-level multiplier for fixed bit-width.

Question: For all possible $a_i, b_i \in \mathbb{B}$:

$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$

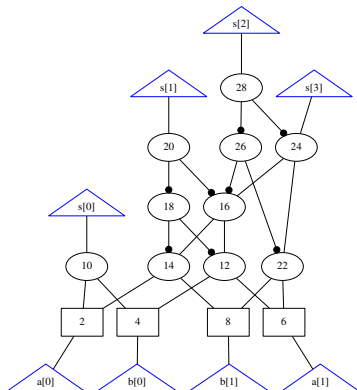


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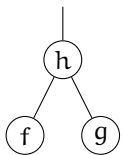
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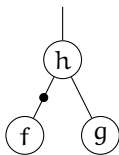


And-Inverter Graph

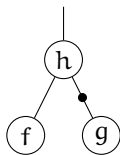
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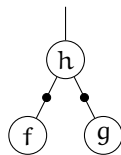
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$$h = \neg f \wedge g$$

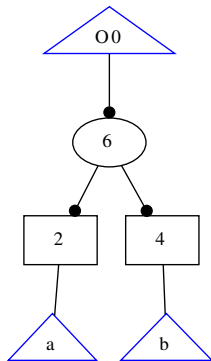


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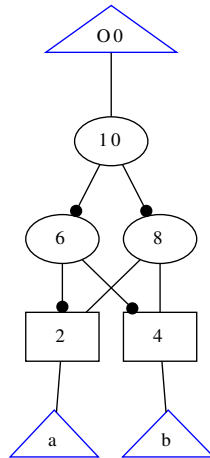


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And-Inverter Graph



$$a \vee b \equiv \neg(\neg a \wedge \neg b)$$



$$a \otimes b \equiv \neg(\neg a \wedge \neg b) \wedge \neg(a \wedge b)$$

And-Inverter Graph

aig 14 4 0 4 10

2

4

6

8

10

20

28

24

10 6 2

12 6 4

14 8 2

16 14 12

18 15 13

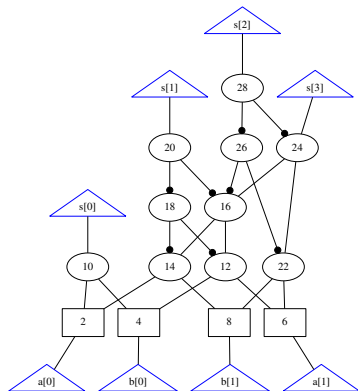
20 19 17

22 8 4

24 22 16

26 23 17

28 27 25

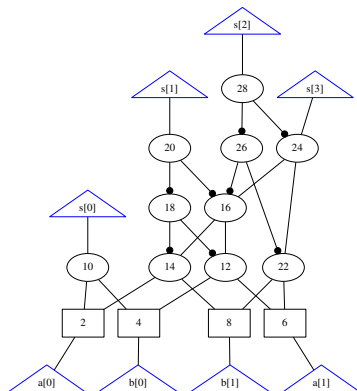


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Circuit Verification

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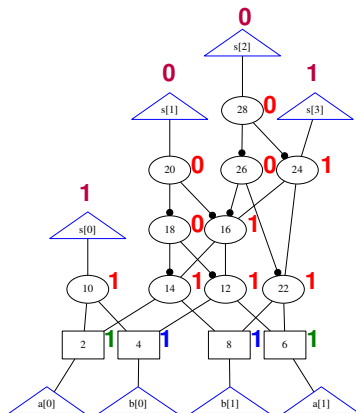


And-Inverter Graph

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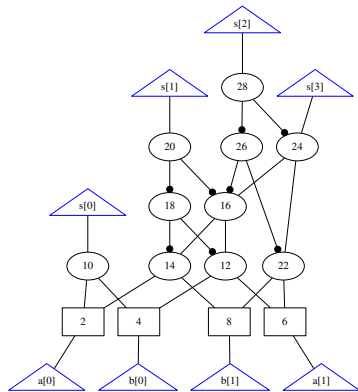
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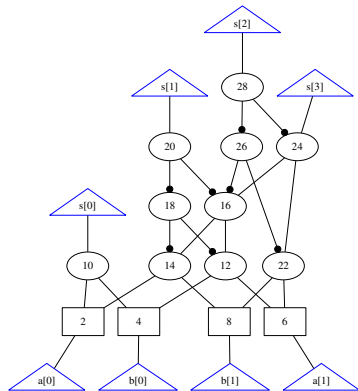
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And-Inverter Graph

Simulation

$a_1 a_0$	$\cdot$	$b_1 b_0$
00	$\cdot$	00
00	$\cdot$	01
00	$\cdot$	10
00	$\cdot$	11
01	$\cdot$	00
01	$\cdot$	01
01	$\cdot$	10
01	$\cdot$	11
10	$\cdot$	00
10	$\cdot$	01
10	$\cdot$	10
10	$\cdot$	11
11	$\cdot$	00
11	$\cdot$	01
11	$\cdot$	10
11	$\cdot$	11

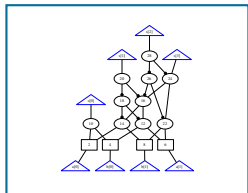


Simulation

Bit-width	Cases
2	16
3	64
4	256
5	1 024
6	4 096
7	16 384
8	65 536
:	:
:	:
32	18 446 744 073 709 551 616
:	:
:	:
64	340 282 366 920 938 463 463 374 607 431 768 211 456

Formal Verification

System



Specification

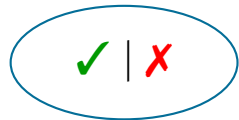
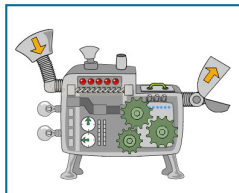
For all $a_i, b_i \in \mathbb{B}$:

$$(2a_1 + a_0) * (2b_1 + b_0)$$
$$= 8s_3 + 4s_2 + 2s_1 + s_0?$$

Mathematical Model

$$B = \{$$
$$x = a_0 * b_0,$$
$$y = a_1 * b_1,$$
$$s_0 = x * y,$$
$$\dots$$
$$\}$$

Automated Decision Process



Formal Verification Techniques

Decision Diagrams

- First technique to detect Pentium bug

[ChenBryant DAC'95]

- Requires knowledge of the layout

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Theorem Proving

- Used in industry, e.g., ACL2
[TemelSlobodovaHunt CAV'20]
- Automated on the RTL level
[Temel TACAS'24]

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Satisfiability Checking (SAT)

- SAT 2016: Exponential run-time of solvers [Biere SATComp'16]
- SAT 2024: Equivalence checking of structural similar circuits
[BiereFallerFazekasFleuryFroleyksPollitt SATComp'24]

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Theorem Proving

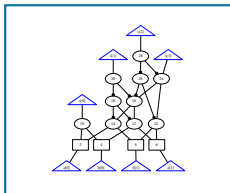
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Algebraic Approach

- Seminal work: [LvKallaEnescu TCAD'13, CiesielskiYuBrownLiuRossi DAC'15]
[SayedGroßeKühneSoekenDrechsler DATE'16]
- Polynomial encoding
- Works for non-trivial multiplier designs

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$B = \{$
 $x = a_0 * b_0,$
 $y = a_1 * b_1,$
 $s_0 = x * y,$
 $\dots$
 $\}$



Specification

$$\sum_{i=0}^{2n-1} 2^i s_i =$$
$$\left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$



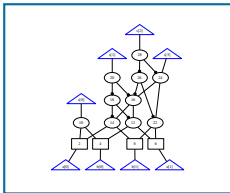
Implication

$= 0$ ✓

$\neq 0$ ✗

Basic Idea of Algebraic Approach

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Polynomials

$B = \{$
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Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$



Ideal Membership

$= 0$ ✓

$\neq 0$ ✗

Multiplier Specification

Unsigned Integers:

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right) \in \mathbb{Q}[X]$$

Multiplier Specification

Unsigned Integers:

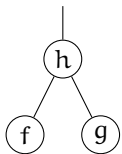
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Multiplier Specification

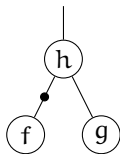
Unsigned Integers:

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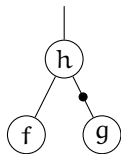
From AIGs to Polynomials



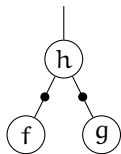
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$$h = \neg f \wedge g$$

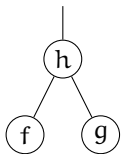


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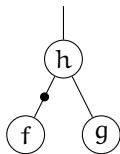
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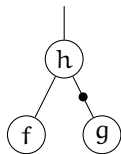
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f	g	h
0	0	0
0	1	0
1	0	0
1	1	1



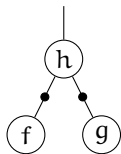
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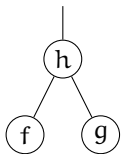
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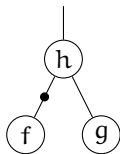
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0	1	0
1	0	0
1	1	1

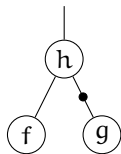
$$-h + fg$$



$$h = \neg f \wedge g$$

f	g	h
0	0	0
0	1	1
1	0	0
1	1	0

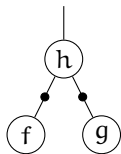
$$-h - fg + g$$



$$h = f \wedge \neg g$$

f	g	h
0	0	0
0	1	0
1	0	1
1	1	0

$$-h - fg + f$$



$$h = \neg f \wedge \neg g$$

f	g	h
0	0	1
0	1	0
1	0	0
1	1	0

$$-h + fg - f - g + 1$$

From AIGs to Polynomials

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2^n}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1 b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

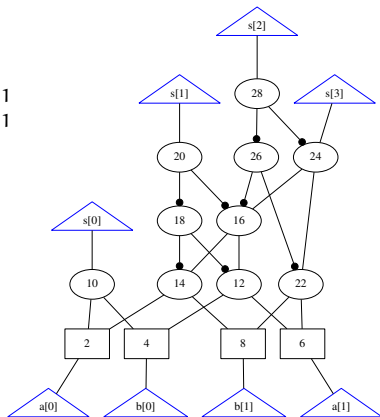
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From AIGs to Polynomials

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

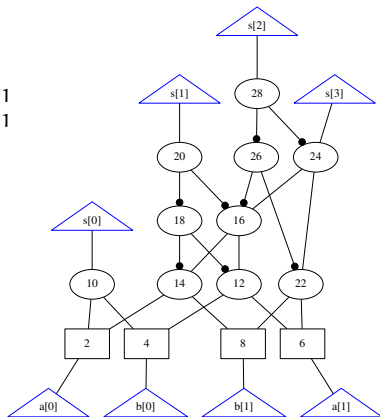
$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
$-s_2 + l_{28}$	$-l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1$
$-s_1 + l_{20}$	$-l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
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Boolean input constraints $B(C) \subseteq \mathbb{Z}^{2n}[X]$.

$a_1, a_0 \in \mathbb{B}$	$-a_1^2 + a_1, -a_0^2 + a_0,$
$b_1, b_0 \in \mathbb{B}$	$-b_1^2 + b_1, -b_0^2 + b_0$

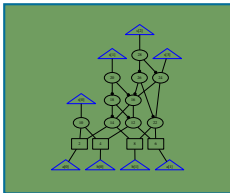
Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1 a_1 - 2b_1 a_0 - 2b_0 a_1 - b_0 a_0$$



Basic Idea of Algebraic Approach

Multiplier



Polynomials

$B = \{$
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Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$

Ideal Membership

$= 0$ ✓
 $\neq 0$ ✗

COMPUTER ALGEBRA



COMPUTER ALGEBRA

IDEALS AND GRÖBNER BASES

Ideal

Definition: Ideal

A subset $I \subset \mathbb{K}[X]$ is an ideal if it satisfies:

- $0 \in I$
- If $f, g \in I$, then $f + g \in I$.
- If $f \in I$ and $h \in \mathbb{K}[X]$ then $hf \in I$.

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Ideal Generated by Finitely Many Polynomials

Let $f_1, \dots, f_s \in \mathbb{K}[X]$. Then we set

$$\langle f_1, \dots, f_s \rangle = \{h_1 f_1 + \dots + h_s f_s \mid h_1, \dots, h_s \in \mathbb{K}[X]\}.$$

$\langle f_1, \dots, f_s \rangle$ is an ideal and is called the ideal generated by $f_1, \dots, f_s$.

We can think of $\langle f_1, \dots, f_s \rangle$ as consisting of all “polynomial consequences” of the equations $f_1 = f_2 = \dots = f_s = 0$.

Applications of Ideals

The Ideal Membership Problem

Given $f \in \mathbb{K}[X]$ and an ideal $I = \langle f_1, \dots, f_s \rangle \subset \mathbb{K}[X]$, determine if $f \in I$.

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Idea Divide f by $f_1, \dots, f_s$?

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Let $I = \langle x^2 - y, xy - x \rangle \subset \mathbb{Q}[x, y]$.

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Spoiler: Yes, because

$$y \cdot (x^2 - y) - x \cdot (xy - x) = x^2 - y^2$$

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$$x^2 - y^2 \xrightarrow{xy - x} x^2 - y^2 \quad (\text{no reduction possible})$$

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- 3 Every ideal $I \subseteq \mathbb{K}[X]$ **has** a Gröbner basis G w.r.t. a fixed monomial order $\prec$.
- 4 Given an arbitrary basis, we can **compute** G **(double-exponential)**.

Orderings on the Monomials

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Question How to order non-linear multivariate polynomials in $\mathbb{K}[X]$?

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$\sigma_1 \prec_{\text{lex}} \sigma_2$ iff there exists an index i with $u_j = v_j$ for all $j < i$, and $u_i < v_i$.

Example: $x_1^2 x_2^7 \prec_{\text{lex}} x_1^3 x_2 \prec_{\text{lex}} x_1^3 x_2^6$

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$\sigma_1 \prec_{\text{dlex}} \sigma_2$ iff either $|\sigma_1| < |\sigma_2|$ or $|\sigma_1| = |\sigma_2|$ and $\sigma_1 \prec_{\text{lex}} \sigma_2$.

Example: $x_1^3 x_2 \prec_{\text{dlex}} x_1^2 x_2^7 \prec_{\text{dlex}} x_1^3 x_2^6$

Computing a Gröbner Basis

Algorithm: Buchberger's Algorithm

Input : $F = \{f_1, \dots, f_s\}$, monomial ordering $<$

Output: Gröbner basis $G = \{g_1, \dots, g_t\}$ w.r.t. $<$, such that $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_t \rangle$

$G = F$;

$C = \{\{g_1, g_2\} \mid g_1, g_2 \in G, g_1 \neq g_2\}$;

while not all pairs $\{g_1, g_2\} \in C$ are marked **do**

 choose unmarked pair $\{g_1, g_2\}$;

 mark $\{g_1, g_2\}$;

$h = \text{normalform of } \text{spol}(g_1, g_2) \text{ w.r.t. } G \quad (\text{spol}(g_1, g_2) \xrightarrow{G} h)$;

if $h \neq 0$ **then**

$C = C \cup \{\{g, h\} \mid g \in G\}$;

$G = G \cup \{h\}$;

return G

S-polynomial. For $p, q \in \mathbb{K}[X]$ with $L = \text{lcm}(\text{lm}(p), \text{lm}(q))$: $\text{spol}(p, q) = \frac{L}{\text{lt}(p)} p - \frac{L}{\text{lt}(q)} q$

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Product Criterion. If $p, q \in \mathbb{K}[x_1, \dots, x_n] \setminus \{0\}$ are such that the leading monomials are coprime, i.e., $\text{lcm}(\text{lm}(p), \text{lm}(q)) = \text{lm}(p) \text{lm}(q)$, then $\text{spol}(p, q)$ reduces to zero mod $\{p, q\}$.

Computing a Gröbner Basis - Example

Recall $I = \langle x^2 - y, xy - x \rangle \subset \mathbb{Q}[x, y]$, order $\prec_{\text{lex}}$ with $x > y$.

Set $f_1 = x^2 - y$, $f_2 = xy - x$, $G = \{f_1, f_2\}$.

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Reduce w.r.t. G : $x^2 - y^2 \xrightarrow{f_1} y^2 - y \neq 0$

$\Rightarrow$ add $f_3 = y^2 - y$ to G . $G = \{f_1, f_2, f_3\}$.

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$\Rightarrow$ add $f_3 = y^2 - y$ to G . $G = \{f_1, f_2, f_3\}$.

Remaining pairs. $\text{spol}(f_1, f_3) \xrightarrow{G} 0$, $\text{spol}(f_2, f_3) \xrightarrow{G} 0$

$\Rightarrow G = \{x^2 - y, xy - x, y^2 - y\}$ is a Gröbner basis of I .

Ideal Membership - Revisited

Recall $f = x^2 - y^2$ and the Gröbner basis $G = \{x^2 - y, xy - x, y^2 - y\}$.

$$x^2 - y^2 \xrightarrow{x^2 - y} -y^2 + y \xrightarrow{y^2 - y} 0$$

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$$x^2 - y^2 \xrightarrow{x^2 - y} -y^2 + y \xrightarrow{y^2 - y} 0$$

The remainder of f w.r.t. the Gröbner basis G is 0.

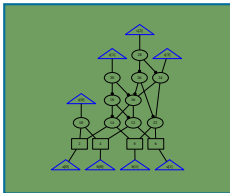
$\Rightarrow f \in I$

BACK TO CIRCUITS



Basic Idea of Algebraic Approach

Multiplier



Polynomials

$B = \{$
 $x - a_0 * b_0,$
 $y - a_1 * b_1,$
 $s_0 - x * y,$
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Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$

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Ideal Membership Problem

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 - Boolean input constraints $B(C)$

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Definition: Multiplier

A circuit C is called a **multiplier** if

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right) \in I(C).$$

Verification Algorithm

Theorem (Kaufmann, Biere, Kauers FMCAD'17) For any circuit C , the set $G(C) \cup B(C)$ forms a Gröbner basis for $I(C)$ w.r.t. any lexicographic monomial order where “gate inputs $<$ gate output”.

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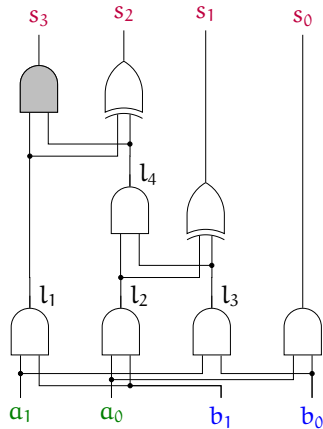
Reduce specification $\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right) \left(\sum_{i=0}^{n-1} 2^i b_i\right)$ by elements of $G(C) \cup B(C)$

until no further reduction is possible, then C is a multiplier iff remainder is zero.

Verification

Gate polynomials $G(C)$.

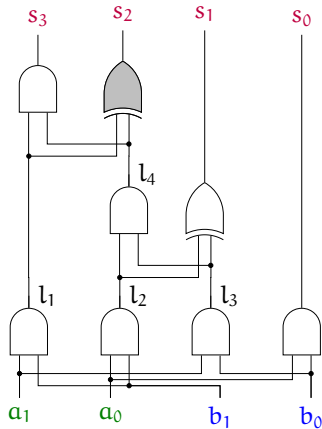
$$s_3 = l_1 \wedge l_4 \quad -s_3 + l_4 l_1,$$



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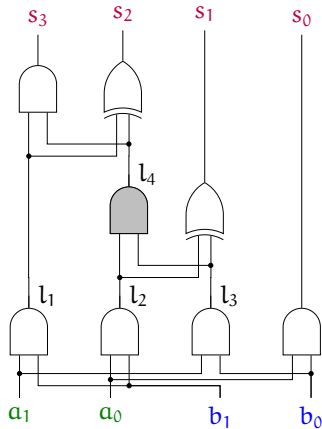
$$\begin{aligned} s_3 &= l_1 \wedge l_4 & -s_3 + l_4 l_1, \\ s_2 &= l_1 \oplus l_4 & -s_2 - 2l_4 l_1 + l_4 + l_1, \end{aligned}$$



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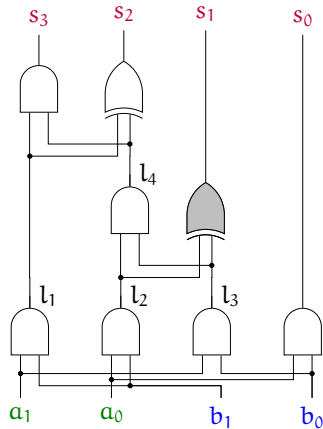
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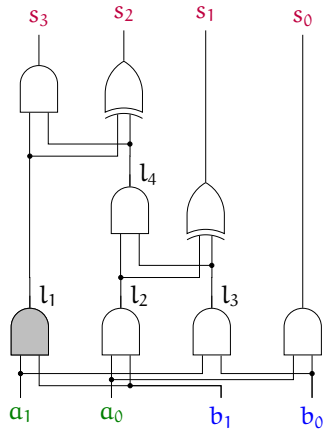
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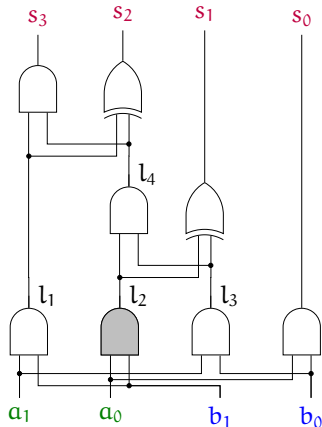
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Verification

Gate polynomials $G(C)$.

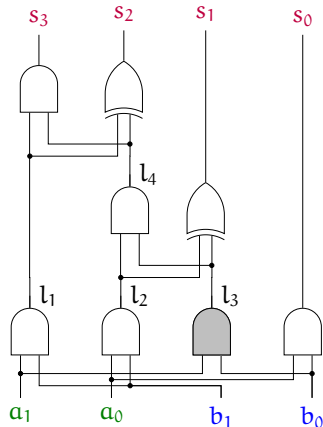
$s_3 = l_1 \wedge l_4$	$-s_3 + l_4 l_1,$
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Gate polynomials $G(C)$.

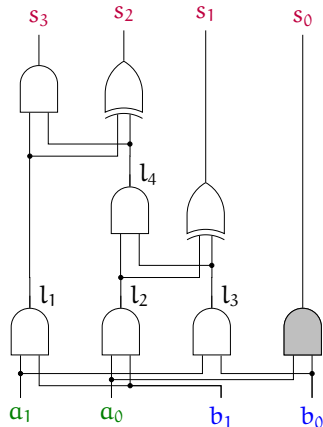
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Verification

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$l_2 = a_0 \wedge b_1$	$-l_2 + a_0 b_1,$
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$s_0 = a_0 \wedge b_0$	$-s_0 + a_0 b_0$



Verification

$$G(C) \cup B(C) = \{$$

$$-s_3 + l_1 l_4,$$

$$-s_2 - 2l_1 l_4 + l_4 + l_1,$$

$$-l_4 + l_2 l_3,$$

$$-s_1 - 2l_2 l_3 + l_3 + l_2,$$

$$-l_1 + a_1 b_1,$$

$$-l_2 + a_0 b_1,$$

$$-l_3 + a_1 b_0,$$

$$-s_0 + a_0 b_0,$$

$$-a_1^2 + a_1,$$

$$-a_0^2 + a_0,$$

$$-b_1^2 + b_1,$$

$$-b_0^2 + b_0\}$$

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

Verification

$$G(C) \cup B(C) = \{$$

$$-s_3 + l_1 l_4,$$

$$-s_2 - 2l_1 l_4 + l_4 + l_1,$$

$$-l_4 + l_2 l_3,$$

$$-s_1 - 2l_2 l_3 + l_3 + l_2,$$

$$-l_1 + a_1 b_1,$$

$$-l_2 + a_0 b_1,$$

$$-l_3 + a_1 b_0,$$

$$-s_0 + a_0 b_0,$$

$$-a_1^2 + a_1,$$

$$-a_0^2 + a_0,$$

$$-b_1^2 + b_1,$$

$$-b_0^2 + b_0\}$$

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$$8l_1 l_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$-l_1 + a_1 b_1,$$

$$-l_2 + a_0 b_1,$$

$$-l_3 + a_1 b_0,$$

$$-s_0 + a_0 b_0,$$

$$-a_1^2 + a_1,$$

$$-a_0^2 + a_0,$$

$$-b_1^2 + b_1,$$

$$-b_0^2 + b_0\}$$

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$$-l_1 + a_1 b_1,$$

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$$-l_3 + a_1 b_0,$$

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$$-l_2 + a_0 b_1,$$

$$-l_3 + a_1 b_0,$$

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$$8l_1 l_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$4l_1 + 2l_3 + 2l_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$-l_1 + a_1 b_1,$$

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$$-l_3 + a_1 b_0,$$

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$$4l_1 + 2l_3 + 2l_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$4l_1 + 2l_3 + 2l_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + s_0 - 2a_1 b_0 - a_0 b_0$$

$$s_0 - a_0 b_0$$

$$0$$

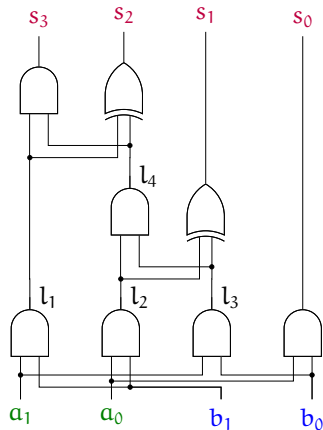
Faulty multiplier

Gate polynomials $G(C)$.

$$\begin{array}{ll}
 s_3 = l_1 \wedge l_4 & -s_3 + l_4 l_1, \\
 s_2 = l_1 \oplus l_4 & -s_2 - 2l_4 l_1 + l_4 + l_1, \\
 l_4 = l_2 \wedge l_3 & -l_4 + l_2 l_3, \\
 s_1 = l_2 \oplus l_3 & -s_1 - 2l_2 l_3 + l_2 + l_3, \\
 l_1 = a_1 \wedge b_1 & -l_1 + a_1 b_1, \\
 l_2 = a_0 \wedge b_1 & -l_2 + a_0 b_1, \\
 l_3 = a_1 \wedge b_0 & -l_3 + a_1 b_0, \\
 s_0 = a_0 \wedge b_0 & -s_0 + a_0 b_0
 \end{array}$$

Boolean input constraints $B(C)$.

$$\begin{array}{ll}
 a_1, a_0 \in \mathbb{B} & a_1(1 - a_1), a_0(1 - a_0), \\
 b_1, b_0 \in \mathbb{B} & b_1(1 - b_1), b_0(1 - b_0)
 \end{array}$$



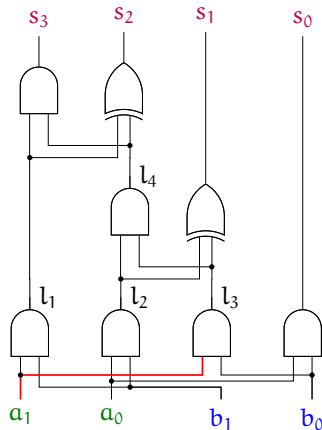
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 s_1 = l_2 \oplus l_3 & -s_1 - 2l_2 l_3 + l_2 + l_3, \\
 l_1 = a_1 \wedge b_1 & -l_1 + a_1 b_1, \\
 l_2 = a_0 \wedge b_1 & -l_2 + a_0 b_1, \\
 l_3 = a_1 \wedge b_0 & -l_3 + a_1 b_0, \\
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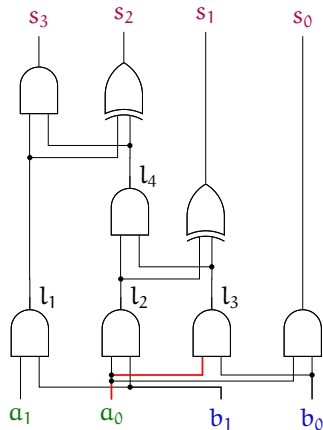
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 s_1 = l_2 \oplus l_3 & -s_1 - 2l_2 l_3 + l_2 + l_3, \\
 l_1 = a_1 \wedge b_1 & -l_1 + a_1 b_1, \\
 l_2 = a_0 \wedge b_1 & -l_2 + a_0 b_1, \\
 l_3 = a_1 \wedge b_0 & -l_3 + a_1 b_0, \\
 s_0 = a_0 \wedge b_0 & -s_0 + a_0 b_0
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 \end{array}$$



Verification of a faulty multiplier

$$G(C) \cup B(C) = \{$$

$$-s_3 + l_1 l_4,$$

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$$-l_4 + l_2 l_3,$$

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$$-l_3 + a_0 b_0,$$

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$$8s_3 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$8l_1 l_4 + 4s_2 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$4l_2 l_3 + 4l_1 + 2s_1 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$4l_1 + 2l_3 + 2l_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$4l_1 + 2l_3 + 2l_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + s_0 - 2a_1 b_0 - a_0 b_0$$

$$s_0 - 2a_1 b_0 + a_0 b_0$$

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$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

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$$4l_1 + 2l_3 + 2l_2 + s_0 - 4a_1 b_1 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + 2l_2 + s_0 - 2a_1 b_0 - 2a_0 b_1 - a_0 b_0$$

$$2l_3 + s_0 - 2a_1 b_0 - a_0 b_0$$

$$s_0 - 2a_1 b_0 + a_0 b_0$$

$$-2a_1 b_0 + 2a_0 b_0$$

Counter Example

Remainder: $-2a_1b_0 + 2a_0b_0 \neq 0$

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$$a_1 = 0, \quad a_0 = 1$$

$$b_1 = 0, \quad b_0 = 1$$

$$01 \cdot 01 = 0001$$

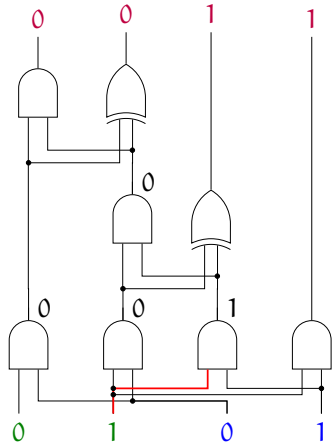
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Verification Algorithm

Theorem (Kaufmann, Biere, Kauers FMCAD'17) For any circuit C , the set $G(C) \cup B(C)$ forms a Gröbner basis for $I(C)$ w.r.t. any lexicographic monomial order where “gate inputs $<$ gate output”.

Reduce specification $\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right) \left(\sum_{i=0}^{n-1} 2^i b_i\right)$ by elements of $G(C) \cup B(C)$ until no further reduction is possible, then C is a multiplier iff remainder is zero.

Verification Algorithm

Theorem (Kaufmann, Biere, Kauers FMCAD'17) For any circuit C , the set $G(C) \cup B(C)$ forms a Gröbner basis for $I(C)$ w.r.t. any lexicographic monomial order where “gate inputs < gate output”.

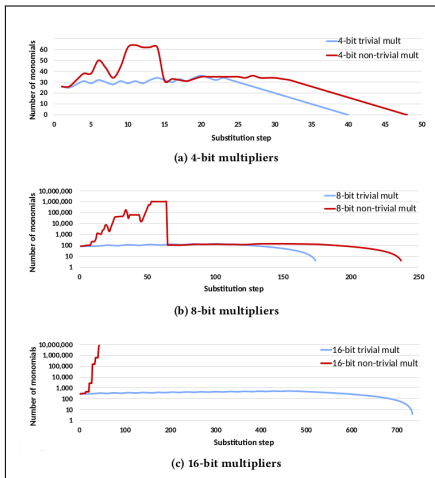
Reduce specification $\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right) \left(\sum_{i=0}^{n-1} 2^i b_i\right)$ by elements of $G(C) \cup B(C)$ until no further reduction is possible, then C is a multiplier iff remainder is zero.

Computational Problems

- 👎 The number of monomials in the intermediate results blows-up.
- 👎 8-bit multiplier cannot be verified within 20 minutes.

Verification Algorithm

[MahzoonGroßeDrechsler ICCAD'18]



Strategies

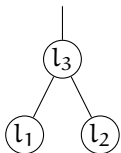
⌞ Encoding

- Embedding negations as literals [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]

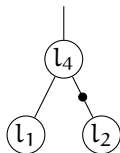
Encoding

[KaufmannBeameBiereNordström DATE'22]

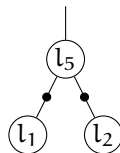
Provide a shorthand notation for inverters.



$$l_3 = l_1 \wedge l_2$$
$$-l_3 + l_1 l_2$$



$$l_4 = l_1 \wedge \neg l_2$$
$$-l_4 - l_1 l_2 + l_1$$



$$l_5 = \neg l_1 \wedge \neg l_2$$
$$-l_5 + l_1 l_2 - l_1 - l_2 + 1$$

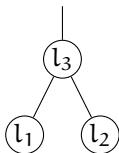
Encoding

[KaufmannBeameBiereNordström DATE'22]

Provide a shorthand notation for inverters.

Dual variables.

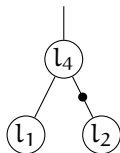
Whenever two variables $l_i, f_i \in \{0, 1\}$ fulfill the relation $f_i = 1 - l_i$, we have $f_i = \text{dual}(l_i)$.



$$l_3 = l_1 \wedge l_2$$

$$-l_3 + l_1 l_2$$

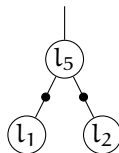
$$-l_3 + l_1 l_2$$



$$l_4 = l_1 \wedge \neg l_2$$

$$-l_4 - l_1 l_2 + l_1$$

$$-l_4 + l_1 f_2$$



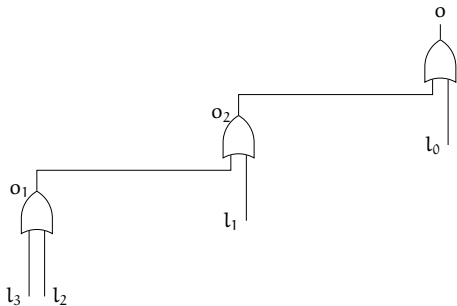
$$l_5 = \neg l_1 \wedge \neg l_2$$

$$-l_5 + l_1 l_2 - l_1 - l_2 + 1$$

$$-l_5 + f_1 f_2$$

OR Gates

$$\begin{aligned}o &= o_2 \vee l_0 & -o + o_2 + l_0 - o_2 l_0, \\o_2 &= o_1 \vee l_1 & -o_2 + o_1 + l_1 - o_1 l_1, \\o_1 &= l_3 \vee l_2 & -o_1 + l_3 + l_2 - l_3 l_2\end{aligned}$$



$$o = l_0 + l_1 - l_0 l_1 + l_2 - l_0 l_2 - l_1 l_2 + l_0 l_1 l_2 + l_3 - l_0 l_3 - l_1 l_3 + l_0 l_1 l_3 - l_2 l_3 + l_0 l_2 l_3 + l_1 l_2 l_3 - l_0 l_1 l_2 l_3$$

$$o = 1 - f_0 f_1 f_2 f_3$$

Strategies

Encoding

- Embedding negations as literals [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]

Preprocessing

- Variable Elimination [MahzoonGroßeDrechsler DAC'19, RitircBiereKauers DATE'18]

Reduction

- Incremental Algorithm [RitircBiereKauers FMCAD'17]
- Dynamic Reduction Order [MahzoonGroßeSchollDrechsler DATE'20, KonradScholl FMCAD'24]

Strategies

Encoding

- Embedding negations as literals [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]

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- Incremental Algorithm [RitircBiereKauers FMCAD'17]
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Demo: d-kfmnn.github.io/circa

Strategies

Encoding

- Embedding negations as literals [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]

Preprocessing

- Variable Elimination [MahzoonGroßeDrechsler DAC'19, RitircBiereKauers DATE'18]

Reduction

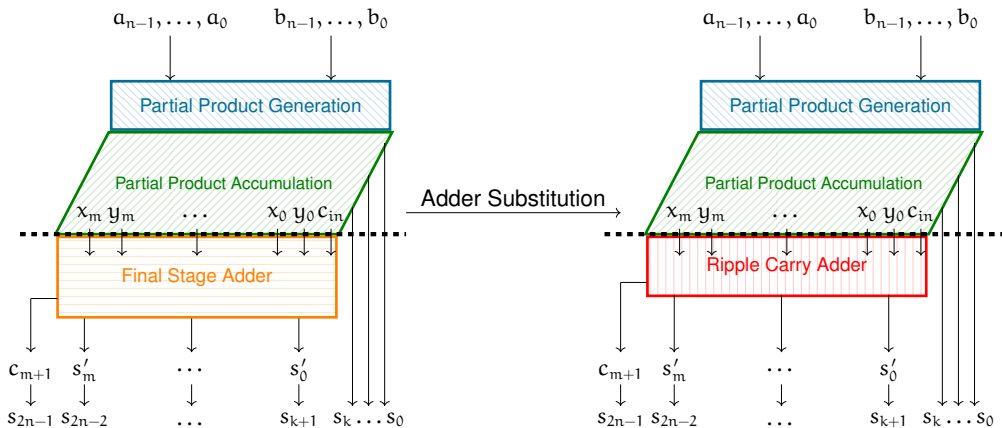
- Incremental Algorithm [RitircBiereKauers FMCAD'17]
- Dynamic Reduction Order [MahzoonGroßeSchollDrechsler DATE'20, KonradScholl FMCAD'24]

Tricky: Final Stage Adder

- Include SAT or BDDs [KaufmannBiereKauers FMCAD'19, DrechslerMahzoon ISEEIE'22]

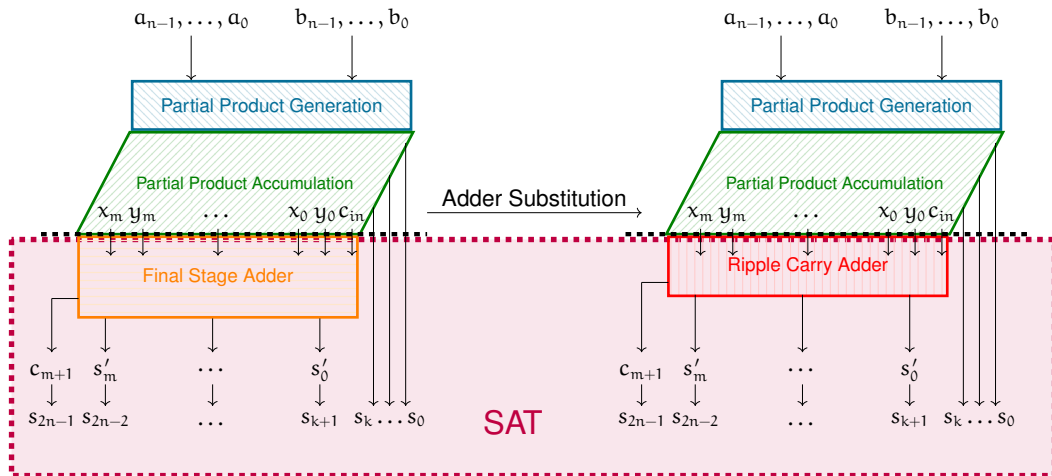
SAT & Computer Algebra

[KaufmannBiereKauers FMCAD'19]



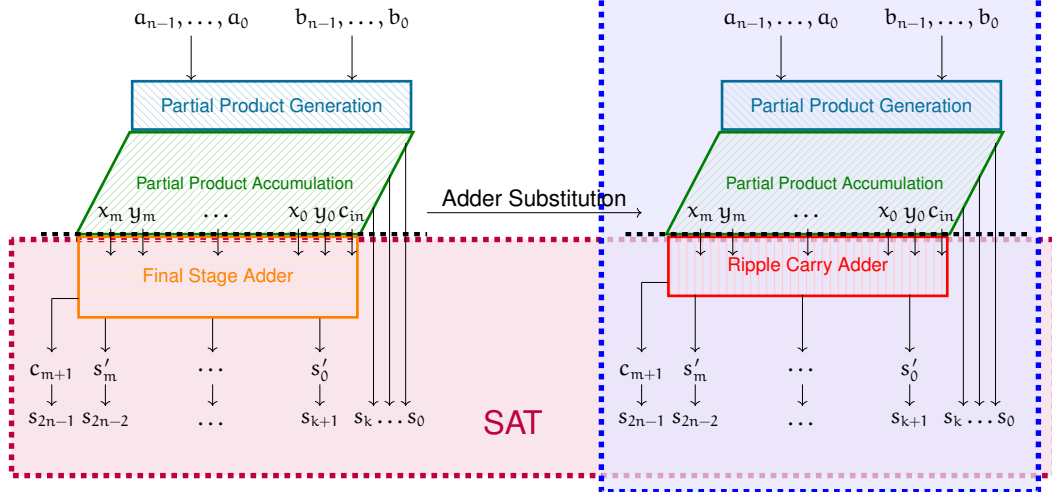
SAT & Computer Algebra

[KaufmannBiereKauers FMCAD'19]



SAT & Computer Algebra

[KaufmannBiereKauers FMCAD'19]



```
danielela@daninovo:~/algprop/src/amulet2$ ./amulet ../benchmarks/btor/btor512.alg -verify -v2
```

```
amulet@danlnovo:~/algprop/src/amulet$ ./amulet ../benchmarks/btor/btor512.aig -verify -v2
[amulet2] AMulet Version 2.0
[amulet2] Aiger multiplier examination tool
[amulet2] Copyright(C) 2020, Daniela Kaufmann, Johannes Kepler University Linz
[amulet2] selected mode: verification
[amulet2] reading '../benchmarks/btor/btor512.aig'
[amulet2] MILOA 2092544 1024 0 1024 2091520
[amulet2] assuming ordering as in BTOR generated benchmarks
[amulet2] (a[0], a[1], ..., a[511])=(input[0], input[2], ..., input[1022])
[amulet2] (b[0], b[1], ..., b[511])=(input[1], input[3], ..., input[1023])
[amulet2] (s[0], ..., s[1023])=(output[0], ..., output[1023])
[amulet2] allocating 2093568 gates
[amulet2] found 262144 partial products
[amulet2] assuming simple pp generator
[amulet2] found 522752 xor-gates
[amulet2] marking xor chain gates
[amulet2] marked 0 xor gates in last slice
[amulet2] remove internal xor gates
```

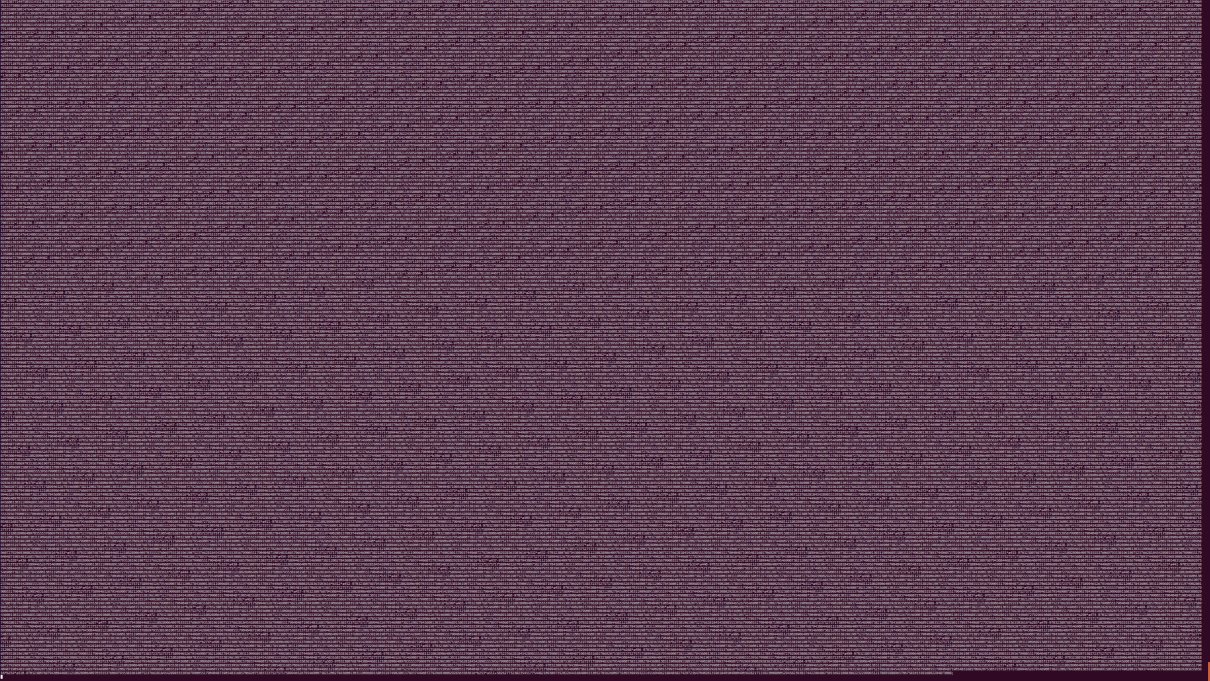
```

[anule2] Aigroprop/src/anule2$ ./anule2 ../benchmarks/btor/btor512.aig -verify -v2
[anule2] Aule2 Version 2.0
[anule2] Aiger multiplier examination tool
[anule2] Copyright(C) 2020, Daniela Kaufmann, Johannes Kepler University Linz
[anule2] selected mode: verification
[anule2] reading '../benchmarks/btor/btor512.aig'
[anule2] MILOA 2092544 1024 0 1024 2091520
[anule2] assuming ordering as in BTOR generated benchmarks
[anule2] (a[0], a[1], ..., a[511]) =(input[0], input[2], ..., input[1022])
[anule2] (b[0], b[1], ..., b[511]) =(input[1], input[3], ..., input[1023])
[anule2] (s[0], ..., s[1023]) =(output[0], ..., output[1023])
[anule2] allocating 2093568 gates
[anule2] found 262144 partial products
[anule2] assuming simple pp generator
[anule2] found 522752 xor-gates
[anule2] marking xor chain gates
[anule2] marked 0 xor gates in last slice
[anule2] remove internal xor gates
[anule2] removed 522752 internal xor gates
[anule2] slicing based on xor
[anule2] remove single occurrence gates
[anule2] removed 0 single occurrence gates
[anule2] moved 0 gates to smaller slices
[anule2] remove gates that are not assigned to slices
[anule2] removed 0 gates that are not assigned to slices
[anule2]
[anule2] started reducing
[anule2] reducing by slice 1023
[anule2] after reducing by slice 1023
[anule2] remainder = -8988465674311579538646525953945123668089884894711532863671504057886633790275048156635423866120376801056005693993569667882939488440720831124642371531973706218888394671243274263815110980062304705972654147604250
28844190753411723144073695655527041361858167525342293149119973622969239858152417678164812112068608;
394884072083112464237153197370621888839467124327426381511098006230470597265414760425028844190753411723144073695655527041361858167525342293149119973622969239858152417678164812112068608;
[anule2]
[anule2] reducing by slice 1022
[anule2] after reducing by slice 1022
[anule2] remainder = -449423283731557897693232629769725618340449424473557664318357520289433168951375240783171193306018840052800284699678483394146974422036041556232118576598685310944419733562163713190755490031152352986327073802125
1442209537670585615720368478277671146574559986811484619929076208839082406056034304*14185072 -44942328373155789769323262976972561834044942447355766431835752028943316895137524078317119330601884005280028469967848339414
69744220360415562321185765986853109444197335621637131907554900311523529863270738021251442209537670585615720368478277635206809290837627671146574559986811484619929076208839082406056034304*14185070+898846567431157953864652595394512366
80898848947115328636715040578866337902750481566354238661203768010560056939935696678829394884407208311246423715319737062188883946712432742638151109800623047059726541476042502884419075341172314407369565527041361858167525534229314911
9973622969239858152417678164812112068608;
[anule2]
[anule2] reducing by slice 1021
[anule2] after reducing by slice 1021
[anule2] remainder = -224711641857789488466163148848628091702247122367788321591787601447165844756876203915885596653009420026400142349839241697073487211018020778116059288299342655472209866781081856595377745015576176493163536901062
5721104768835292807860184239138817603404645418813835573287279993405742309964538104419541203028017152*14185038 -224711641857789488466163148848628091
70224712236778832159178760144716584475687620391588559665300942002640014234983924169707348721101802077811605928829934265547220986678108185659537774501557617649316353690106257211047688352928078601842391388176034046454188138355732872
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350467285294794960107031877163314360505878423580552717416452810213936256441506719861839980217226929893614313258623609084051456;
[anule2]
[anule2] reducing by slice 1020
[anule2] after reducing by slice 1020
[anule2] remainder = -1123558209288947442330815744243140458511235611838941607958938007235829223784381019579427983265047100132000711749196208485367436055090103890580296441496713277361049333905409282976888872507788088246581768450531
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674360550901038905802964414967132773610493339054092829768888725077880882465817684505312860552384417646403930092119569408801702322709406917786643639996702871154982269052209770601514008576*14184990 -112355820928894744233081574424314045
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72677880882465817684505312860552384417646403930092119569408801702322709406917786643639996702871154982269052209770601514008576*14184986+44942328371557897693232629769725618340449424473557664318357520289433168951375240783171193306018
8408528002846996784833941126052420360415562321185765986853109444197335621637131907554900311523529863270738021251442209537670585615720368478277635206809290837627671146574559986811484619929076208839082406056034304;

```

[illegible]







```

[anulet2] remainder is -4096*13040-4096*13038-4096*13036-4096*13034-4096*13032-4096*13030-4096*13028-4096*13026-4096*13024-4096*13022+4096*13016+40960;
[anulet2]
[anulet2] reducing by slice 11
[anulet2] after reducing by slice 11
[anulet2] remainder is -2048*12870-2048*12868-2048*12866-2048*12864-2048*12862-2048*12860-2048*12858-2048*12856-2048*12854+2048*12848+18432;
[anulet2]
[anulet2] reducing by slice 10
[anulet2] after reducing by slice 10
[anulet2] remainder is -1024*12716-1024*12714-1024*12712-1024*12710-1024*12708-1024*12706-1024*12704-1024*12702+1024*12696+8192;
[anulet2]
[anulet2] reducing by slice 9
[anulet2] after reducing by slice 9
[anulet2] remainder is -512*12578-512*12576-512*12574-512*12572-512*12570-512*12568-512*12566+512*12560+3584;
[anulet2]
[anulet2] reducing by slice 8
[anulet2] after reducing by slice 8
[anulet2] remainder is -256*12456-256*12454-256*12452-256*12450-256*12448-256*12446+256*12440+1536;
[anulet2]
[anulet2] reducing by slice 7
[anulet2] after reducing by slice 7
[anulet2] remainder is -128*12350-128*12348-128*12346-128*12344-128*12342+128*12336+640;
[anulet2]
[anulet2] reducing by slice 6
[anulet2] after reducing by slice 6
[anulet2] remainder is -64*12260-64*12258-64*12256-64*12254+64*12248+256;
[anulet2]
[anulet2] reducing by slice 5
[anulet2] after reducing by slice 5
[anulet2] remainder is -32*12186-32*12184-32*12182+32*12176+96;
[anulet2]
[anulet2] reducing by slice 4
[anulet2] after reducing by slice 4
[anulet2] remainder is -16*12128-16*12126+16*12120+32;
[anulet2]
[anulet2] reducing by slice 3
[anulet2] after reducing by slice 3
[anulet2] remainder is -8*12086+8*12080+8;
[anulet2]
[anulet2] reducing by slice 2
[anulet2] after reducing by slice 2
[anulet2] remainder is 4*12056;
[anulet2]
[anulet2] reducing by slice 1
[anulet2] after reducing by slice 1
[anulet2] remainder is 0;
[anulet2]
[anulet2] reducing by slice 0
[anulet2] after reducing by slice 0
[anulet2] remainder is 0;
[anulet2]
[anulet2] CORRECT MULTIPLIER
[anulet2]
[anulet2] maximum resident set size:          1966.98 MB
[anulet2] used time for initializing:          1.33 seconds
[anulet2] used time for slicing/elimination:  7.62 seconds
[anulet2] used time for reduction:           30.72 seconds
[anulet2] used time for freeing memory:      3.40 seconds
[anulet2] total process time:               43.07 seconds
dante1a@dantinovo:~/algprop/src/anulet2$

```

Strategies

Encoding

- Embedding negations as literals [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]

Preprocessing

- Variable Elimination [MahzoonGroßeDrechsler DAC'19, RitircBiereKauers DATE'18]

Reduction

- Incremental Algorithm [RitircBiereKauers FMCAD'17]
- Dynamic Reduction Order [MahzoonGroßeSchollDrechsler DATE'20, KonradScholl FMCAD'24]

Tricky: OR Gates in final stage adder

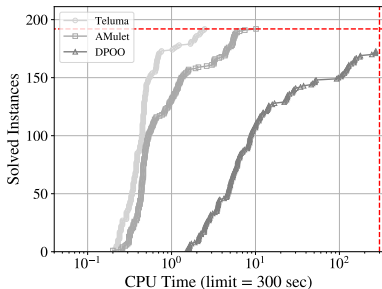
- Include SAT or BDDs [KaufmannBiereKauers FMCAD'19, DrechslerMahzoon ISEEIE'22]

All of these strategies rely on a **lexicographic term ordering** and aim to find an optimal reduction order.

Strategies

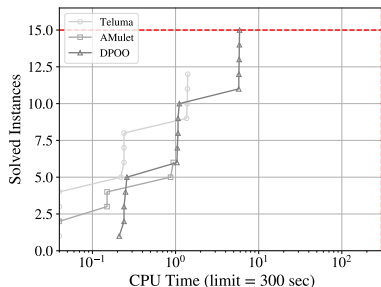
Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Optimized multipliers

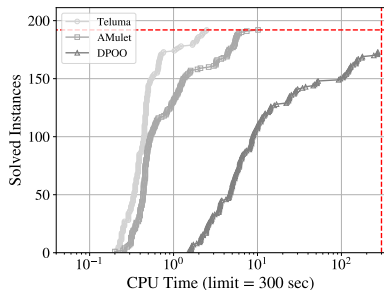
- 5 optimization scripts
- for 32, 64, 128 bit inputs



Strategies

Unoptimized multipliers

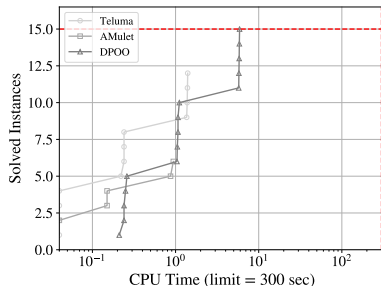
- 192 architectures
- 64 bit inputs



👎 Fast approaches are overfitted

Optimized multipliers

- 5 optimization scripts
- for 32, 64, 128 bit inputs



A New Idea

💡 Change monomial order to **degree-lexicographic order** (Kaufmann, Berthomieu TACAS'25)

	$\prec_{\text{lex}}$	$\prec_{\text{dlex}}$
GB Computation	✓ Easy	⚠ Hard
Spec Reduction	⚠ Hard	✓ Easy

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	$\prec_{\text{lex}}$	$\prec_{\text{dlex}}$
GB Computation	✓ Easy	⚠ Hard
Spec Reduction	⚠ Hard	✓ Easy

Theorem (Kaufmann, Berthomieu TACAS'25) If the specification polynomial is linear, then the linear polynomials in a $\prec_{\text{dlex}}$ -GB suffice to verify the circuit.

Verification via Linearization

Input: Circuit C , specification polynomial S

Output: Determine whether C fulfils S

- 1 $G \leftarrow G(C) \cup B(C)$
- 2 $S_{\text{lin}}, G \leftarrow \text{Linearize-Spec}(S, G)$
- 3 $L \leftarrow \text{LINEAR-POLIES-OF-}\prec_{\text{dlex}}\text{-GB}(G)$
- 4 $S_{\text{lin}} \leftarrow \text{Linear-Reduce}(S_{\text{lin}}, L)$
- 5 **return** $S_{\text{lin}} \stackrel{?}{=} 0$

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$$S_{\text{lin}}, G_{\text{ext}} \leftarrow \text{Linearize-Spec}(S, G)$$

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1 b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

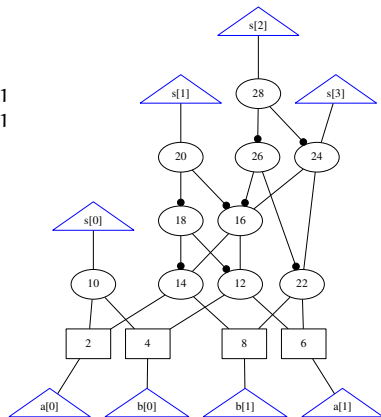
$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0 b_1$$

$$-l_{12} + a_1 b_0$$

$$-l_{10} + a_0 b_0$$



Specification $S \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1a_1 - 2b_1a_0 - 2b_0a_1 - b_0a_0$$

$$S_{\text{lin}}, G_{\text{ext}} \leftarrow \text{Linearize-Spec}(S, G)$$

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

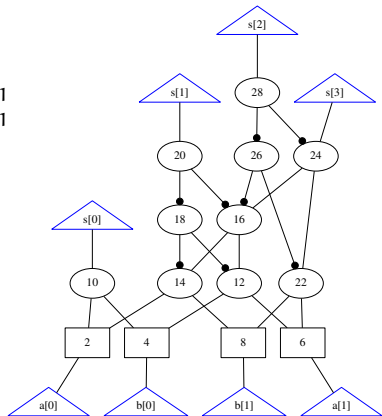
$$\begin{array}{ll} -s_3 + l_{24} & -l_{22} + a_1 b_1 \\ -s_2 + l_{28} & -l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1 \\ -s_1 + l_{20} & -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\ -s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\ -l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\ -l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\ -l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0 \end{array}$$

Extension polynomials $G_{\text{ext}} \subseteq \mathbb{Z}^{2n}[X]$.

$$\begin{array}{ll} -t_{11} + a_1 b_1 & -t_{01} + a_0 b_1 \\ -t_{10} + a_1 b_0 & -t_{00} + a_0 b_0 \end{array}$$

Linear Specification $S_{\text{lin}} \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4t_{11} - 2t_{10} - 2t_{01} - t_{00}$$



$$S_{\text{lin}}, G_{\text{ext}} \leftarrow \text{Linearize-Spec}(S, G)$$

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

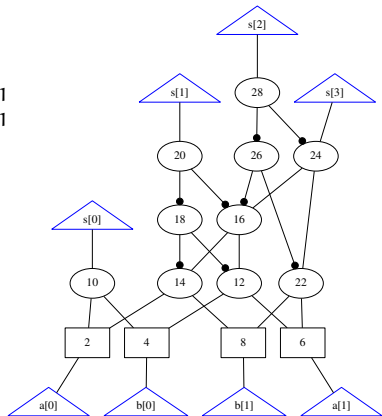
$$\begin{array}{ll} -s_3 + l_{24} & -l_{22} + a_1 b_1 \\ -s_2 + l_{28} & -l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1 \\ -s_1 + l_{20} & -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\ -s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\ -l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\ -l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\ -l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0 \end{array}$$

Extension polynomials $G_{\text{ext}} \subseteq \mathbb{Z}^{2n}[X]$.

$$\begin{array}{ll} -t_{11} + a_1 b_1 & -t_{01} + a_0 b_1 \\ -t_{10} + a_1 b_0 & -t_{00} + a_0 b_0 \end{array}$$

Linear Specification $S_{\text{lin}} \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4t_{11} - 2t_{10} - 2t_{01} - t_{00}$$



$S_{\text{lin}}, G_{\text{ext}} \leftarrow \text{Linearize-Spec}(S, G)$

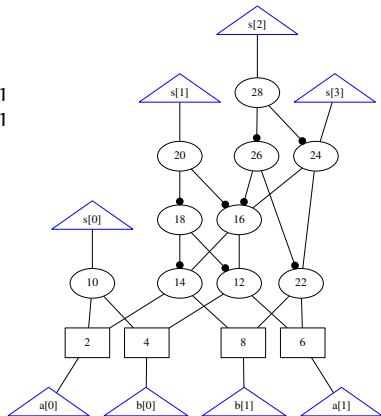
Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
$-s_2 + l_{28}$	$-l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1$
$-s_1 + l_{20}$	$-l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
$-l_{24} + l_{22} l_{16}$	$-l_{10} + a_0 b_0$

Extension polynomials $G_{\text{ext}} \subseteq \mathbb{Z}^{2n}[X]$.

Linear Specification $S_{\text{lin}} \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$



Verification via Linearization

Input: Circuit C , specification polynomial S

Output: Determine whether C fulfils S

- 1 $G \leftarrow G(C) \cup B(C)$
- 2 $S_{\text{lin}}, G \leftarrow \text{Linearize-Spec}(S, G)$
- 3 $L \leftarrow \text{LINEAR-POLIES-OF-}\prec_{\text{dlex}}\text{-GB}(G)$
- 4 $S_{\text{lin}} \leftarrow \text{Linear-Reduce}(S_{\text{lin}}, L)$
- 5 **return** $S_{\text{lin}} \stackrel{?}{=} 0$

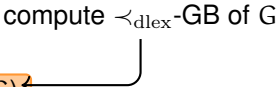
Verification via Linearization

Input: Circuit C , specification polynomial S

Output: Determine whether C fulfils S

```
1  $G \leftarrow G(C) \cup B(C)$ 
2  $S_{\text{lin}}, G \leftarrow \text{Linearize-Spec}(S, G)$ 
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4  $S_{\text{lin}} \leftarrow \text{Linear-Reduce}(S_{\text{lin}}, L)$ 
5 return  $S_{\text{lin}} \stackrel{?}{=} 0$ 
```

compute $\prec_{\text{dlex}}\text{-GB}$ of G



Verification via Linearization

Input: Circuit C , specification polynomial S

Output: Determine whether C fulfils S

```
1  $G \leftarrow G(C) \cup B(C)$ 
2  $S_{\text{lin}}, G \leftarrow \text{Linearize-Spec}(S, G)$ 
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5 return  $S_{\text{lin}} \stackrel{?}{=} 0$ 
```

compute $\prec_{\text{dlex}}\text{-GB}$ of G



Computing a $\prec_{\text{dlex}}\text{-GB}$ for the whole circuit is infeasible.

Verification via Linearization

Input: Circuit C , specification polynomial S

Output: Determine whether C fulfils S

```
1  $G \leftarrow G(C) \cup B(C)$ 
2  $S_{\text{lin}}, G \leftarrow \text{Linearize-Spec}(S, G)$ 
3  $L \leftarrow \text{LINEAR-POLIES-OF-}\prec_{\text{dlex}}\text{-GB}(G)$ 
4  $S_{\text{lin}} \leftarrow \text{Linear-Reduce}(S_{\text{lin}}, L)$ 
5 return  $S_{\text{lin}} \stackrel{?}{=} 0$ 
```

compute $\prec_{\text{dlex}}\text{-GB}$ of $G' \subseteq G$

Computing a $\prec_{\text{dlex}}\text{-GB}$ for the whole circuit is infeasible.

$\rightsquigarrow$ **work locally:** extract sub-circuits and only compute $\prec_{\text{dlex}}\text{-GBs}$ for them

Local Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1 b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

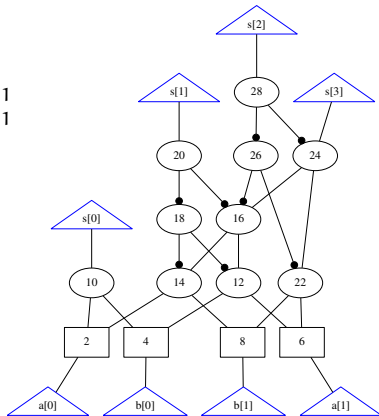
$$-l_{14} + a_0 b_1$$

$$-l_{12} + a_1 b_0$$

$$-l_{10} + a_0 b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$



Local Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1 b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0 b_1$$

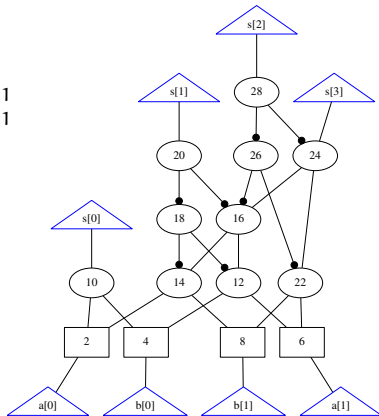
$$-l_{12} + a_1 b_0$$

$$-l_{10} + a_0 b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$



Local Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$-s_3 + l_{24}$$

$$-s_2 + l_{28}$$

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$$-s_0 + l_{10}$$

$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1 b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0 b_1$$

$$-l_{12} + a_1 b_0$$

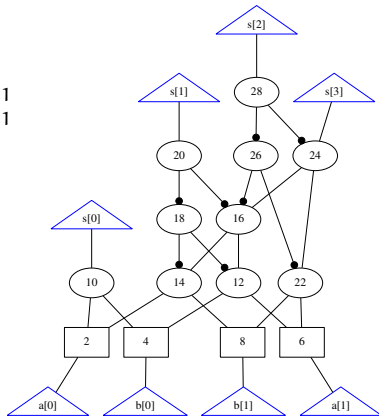
$$-l_{10} + a_0 b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



Local Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$-s_3 + l_{24}$$

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$$-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

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$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0 b_1$$

$$-l_{12} + a_1 b_0$$

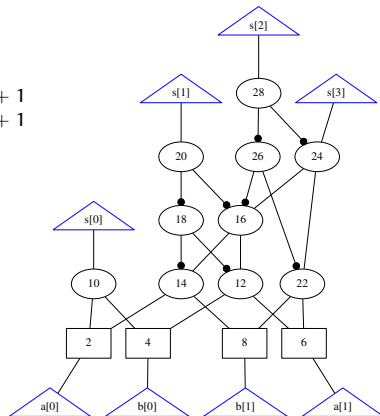
$$-l_{10} + a_0 b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$

$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



Local Linearization

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$$-s_3 + l_{24}$$

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$$-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1$$

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$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} +$$

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$$-l_{16} + l_{14}l_{12}$$

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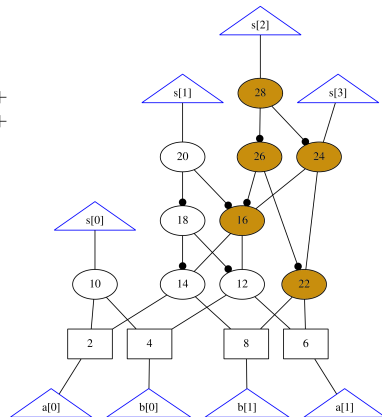
$$-l_{10} + a_0 b_0$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

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$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



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$$-s_2 + l_{28}$$

$$-s_1 + l_{20}$$

$$-s_0 + l_{10}$$

$$-l_{28} - l_{26} - l_{24} + 1$$

$$-l_{26} + l_{24} - l_{22} - l_{16} + 1$$

$$-l_{24} + l_{22}l_{16}$$

$$-l_{22} + a_1b_1$$

$$-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1$$

$$-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1$$

$$-l_{16} + l_{14}l_{12}$$

$$-l_{14} + a_0b_1$$

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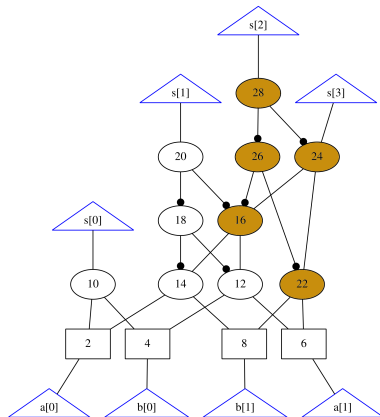
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$$4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12}$$



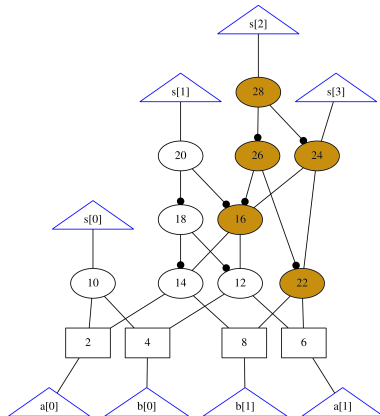
Local Linearization

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$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
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$-s_1 + l_{20}$	$-l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
$-l_{24} + l_{22} l_{16}$	$-l_{10} + a_0 b_0$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$\begin{aligned}
 &8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &\quad 4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &\quad - 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4
 \end{aligned}$$



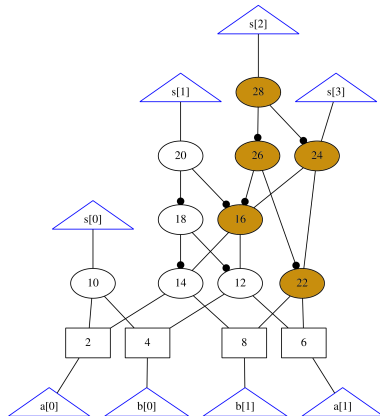
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$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
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Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

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 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &\quad 4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &\quad - 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4 \\
 &\quad 2l_{20} + 4l_{16} - 2l_{14} - 2l_{12}
 \end{aligned}$$



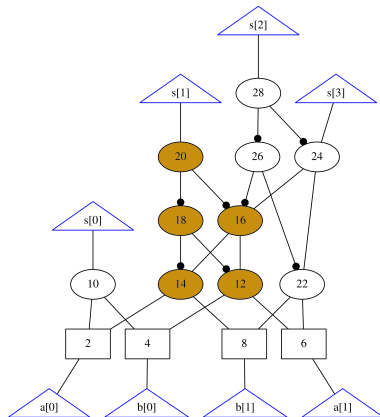
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$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
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Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

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 &\quad 2l_{20} + 4l_{16} - 2l_{14} - 2l_{12}
 \end{aligned}$$



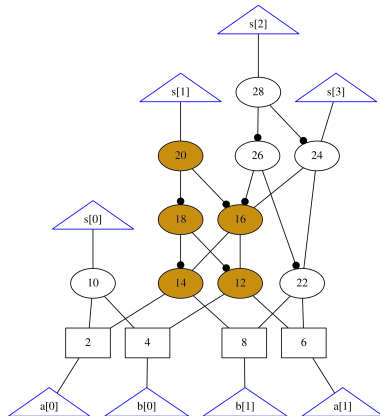
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Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$-s_3 + l_{24}$	$-l_{22} + a_1 b_1$
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$-s_1 + l_{20}$	$-l_{18} + l_{16} - l_{14} - l_{12} + 1$
$-s_0 + l_{10}$	$-l_{16} + l_{14} l_{12}$
$-l_{28} - l_{26} - l_{24} + 1$	$-l_{14} + a_0 b_1$
$-l_{26} + l_{24} - l_{22} - l_{16} + 1$	$-l_{12} + a_1 b_0$
$-l_{24} + l_{22} l_{16}$	$-l_{10} + a_0 b_0$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$\begin{aligned}
 &8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &\quad 4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &\quad - 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4 \\
 &\quad \quad 2l_{20} + 4l_{16} - 2l_{14} - 2l_{12} \\
 &\quad \quad - 2l_{18} + 2l_{16} - 2l_{14} - 2l_{12} + 2
 \end{aligned}$$



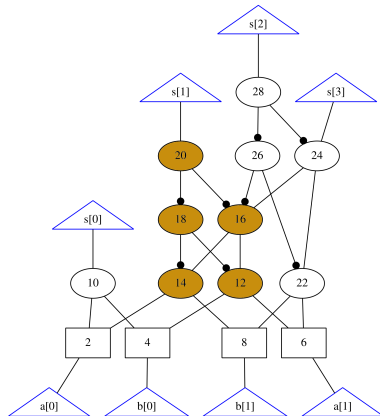
Local Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$\begin{array}{ll}
 -s_3 + l_{24} & -l_{22} + a_1 b_1 \\
 -s_2 + l_{28} & -l_{20} - l_{18} - l_{16} + 1 \\
 -s_1 + l_{20} & -l_{18} + l_{16} - l_{14} - l_{12} + 1 \\
 -s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
 -l_{28} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
 -l_{26} + l_{24} - l_{22} - l_{16} + 1 & -l_{12} + a_1 b_0 \\
 -l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0
 \end{array}$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$\begin{aligned}
 &8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &4s_2 + 2s_1 + s_0 + 8l_{24} - 4l_{22} - 2l_{14} - 2l_{12} - l_{10} \\
 &\quad 4l_{28} + 8l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} \\
 &\quad - 4l_{26} + 4l_{24} - 4l_{22} + 2l_{20} - 2l_{14} - 2l_{12} + 4 \\
 &\quad \quad 2l_{20} + 4l_{16} - 2l_{14} - 2l_{12} \\
 &\quad \quad - 2l_{18} + 2l_{16} - 2l_{14} - 2l_{12} + 2 \\
 &\quad \quad \quad 0
 \end{aligned}$$

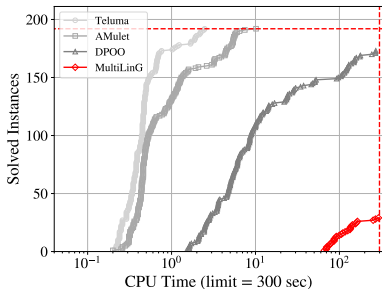


First Approach

(Kaufmann, Berthomieu TACAS'25)

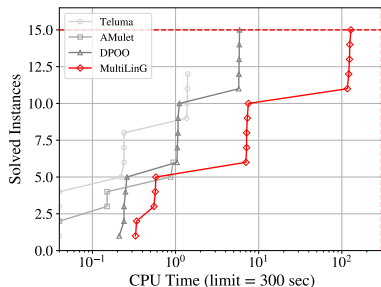
Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Optimized multipliers

- 5 optimization scripts
- for 32, 64, 128 bit inputs

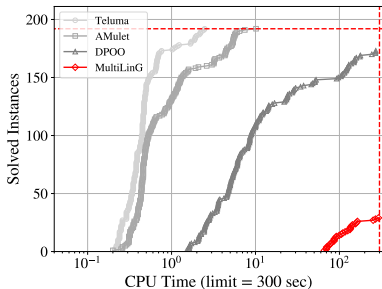


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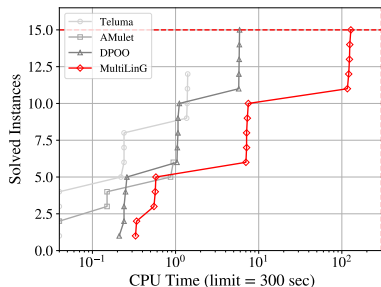
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Approach seems to be more robust against optimization



Black-box GB computation is too costly (can handle ≤ 8 vars)



Completely ignores existing structure

Two Targeted Approaches

(Hofstadler, Kaufmann, CP'25)

FGLM-style Linearization

Guess & Prove Linearization

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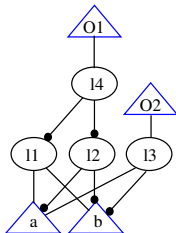
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- 👍 Exploit existing structure $\rightsquigarrow$ a lot faster than GB computation
- 👎 Runtime exponential in number of variables (can handle ≤ 15 vars)

FGLM-style Linearization

Ideal $I \subseteq \mathbb{Q}[a, b, l_1, l_2, l_3, l_4]$ **generated by:**

$$l_1 - ab, \quad l_2 - (1 - a)(1 - b), \quad l_3 - a(1 - b), \quad l_4 - (1 - l_1)(1 - l_2), \quad a^2 - a, \quad b^2 - b$$



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$$l_1 = ab, \quad l_2 = ab - a - b + 1, \quad l_3 = -ab + a, \quad l_4 = -2ab + a + b$$

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2. Build Coefficient Matrix A:

$$\begin{array}{ccccccc} \text{NF}_G(1) & \text{NF}_G(a) & \text{NF}_G(b) & \text{NF}_G(l_1) & \text{NF}_G(l_2) & \text{NF}_G(l_3) & \text{NF}_G(l_4) \\ \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -2 \end{array} \right) & \begin{array}{l} 1 \\ a \\ b \\ ab \end{array} \end{array}$$

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$$\begin{array}{cccccc} 1 & a & b & l_1 & l_2 & l_3 & l_4 \\ \left(\begin{array}{cccccc} 0 & -1 & -1 & 2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & 0 & 0 \end{array} \right) \end{array}$$

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4. Read off Linear Polynomials:

$$l_4 + 2l_1 - a - b, \quad l_3 + l_1 - a, \quad l_2 - l_1 + a + b - 1$$

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Guess & Prove Linearization

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- ✓ our ideals are radical
- ✓ our ideals are zero-dimensional
- ✓ our ideals don't have additional roots in algebraic closure
- ✓ we can efficiently test ideal membership of linear polynomials

Guess & Prove Linearization

Given: $G \subseteq \mathbb{K}[X]$ satisfying the assumptions from before

Compute: $\mathbb{K}$ -VS basis of $\{f \in \langle G \rangle \mid \deg(f) \leq 1\}$

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Algorithm

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Let $G \subseteq \mathbb{Q}[a, b, l_1, l_2, l_3, l_4]$ **consist of:**

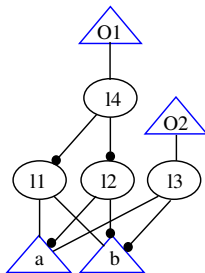
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Kernel yields **Guesses**:

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In this case, SAT solver returns **SAT** and $p = (1, 1, 1, 0, 0, 0)$.

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...how many iterations of the repair loop do we need? ≤ 3

Verification via Linearization

Input: Circuit C , specification polynomial S

Output: Determine whether C fulfils S

```
1  $G \leftarrow G(C) \cup B(C)$ 
2  $S_{\text{lin}}, G \leftarrow \text{Linearize-Spec}(S, G)$ 
3  $L \leftarrow \text{LINEAR-POLIES-OF-}\prec_{\text{dlex}}\text{-GB}(G)$ 
4  $S_{\text{lin}} \leftarrow \text{Linear-Reduce}(S_{\text{lin}}, L)$ 
5 return  $S_{\text{lin}} \stackrel{?}{=} 0$ 
```

compute $\prec_{\text{dlex}}\text{-GB}$ of $G' \subseteq G$

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pick $G' \subseteq G$

if $|G'|$ is small use FGLM

else use Guess & Prove

TALISMAN

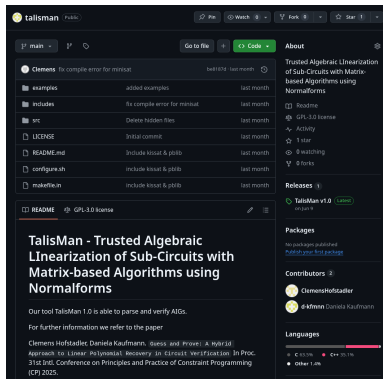
 <https://github.com/d-kfmnn/talisman/>



 Uses FLINT for matrix computations

 Guess and Prove Algorithm:

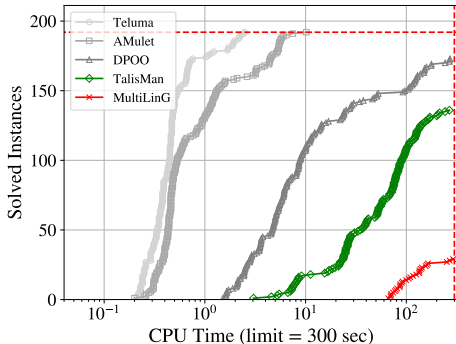
- ↔ Uses PBLIB to translate guessed polynomial to CNF
-  Uses KISSAT as a SAT solver
-  Uses caching of isomorphic subcircuits



Evaluation – Multiplier Circuits

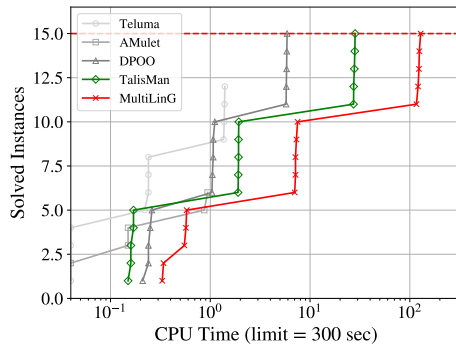
Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Optimized multipliers

- 5 optimization scripts
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Big Coefficients

$$S \rightsquigarrow \dots \rightsquigarrow \sum_{i=1}^D c_i \cdot m_i \rightsquigarrow \dots \rightsquigarrow 0$$

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Extensive prior work:

Encodings, Preprocessing, Dynamic division
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Coefficients

c_i

Values up to $\geq 2^{127}$

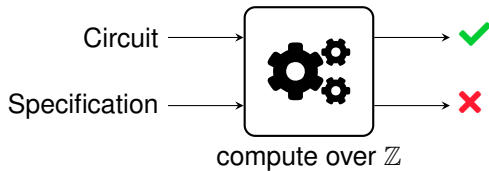
👎 Arbitrary precision arithmetic

👎 CNF Translation explodes

Largely ignored so far!

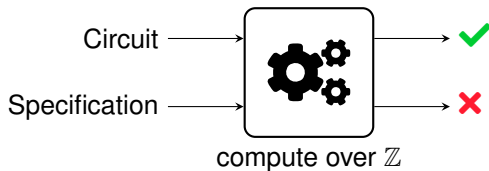
Big Coefficients

Classical Verification



Big Coefficients

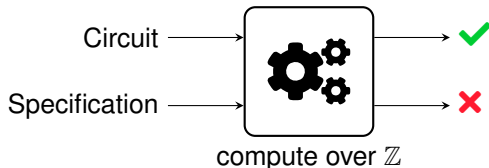
Classical Verification



Specification for a 64-bit multiplier: $\left(\sum_{i=0}^{63} 2^i a_i\right) \cdot \left(\sum_{i=0}^{63} 2^i b_i\right) - \sum_{i=0}^{127} 2^i s_i$

Big Coefficients

Classical Verification

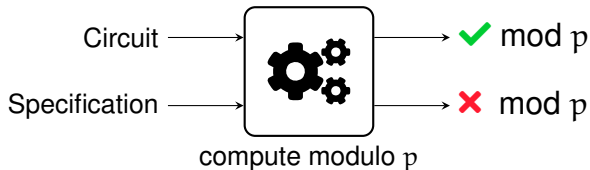


Specification for a 64-bit multiplier: $\left(\sum_{i=0}^{63} 2^i a_i\right) \cdot \left(\sum_{i=0}^{63} 2^i b_i\right) - \sum_{i=0}^{127} 2^i s_i$

Observation: Spec can only take values between -2^{128} and 2^{128} .

Big Coefficients

Modular Verification (Kaufmann, Biere, Kauers FMCAD'19)

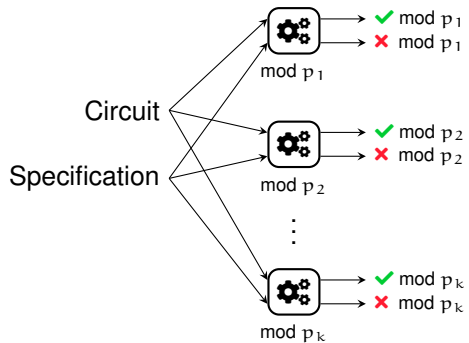


Specification for a 64-bit multiplier: $\left(\sum_{i=0}^{63} 2^i a_i\right) \cdot \left(\sum_{i=0}^{63} 2^i b_i\right) - \sum_{i=0}^{127} 2^i s_i$

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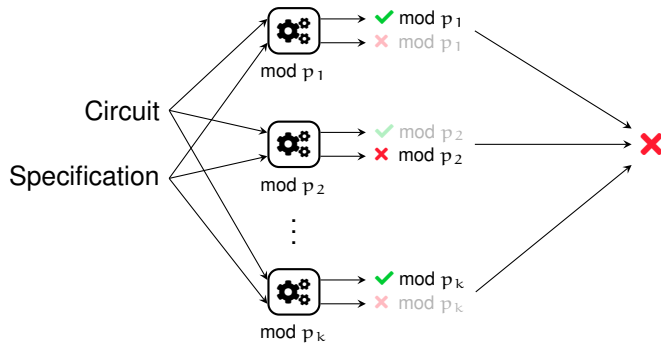
Multimodular Verification

[HofstadlerKaufmannChen IJCAR'26]



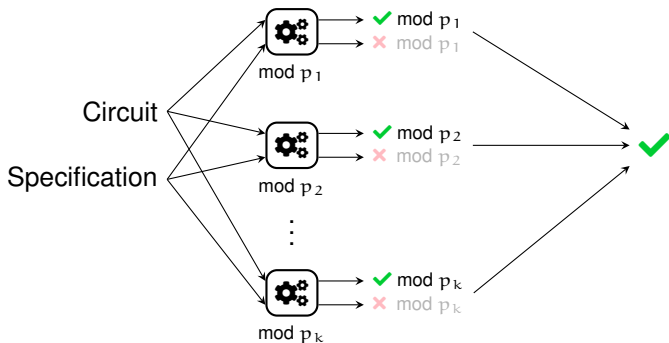
Multimodular Verification

[HofstadlerKaufmannChen IJCAR'26]



Multimodular Verification

[HofstadlerKaufmannChen IJCAR'26]

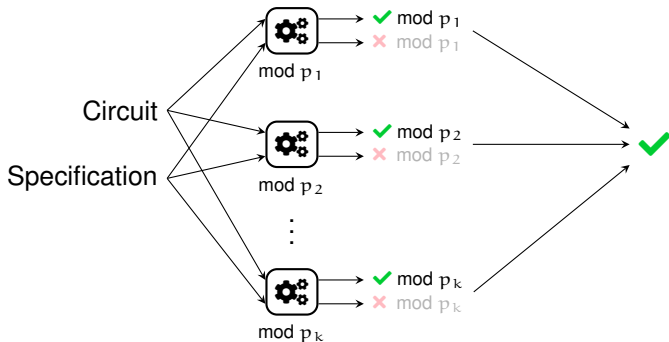


Theorem (Hofstadler, Kaufmann, Chen '26)

If $p_1 p_2 \cdots p_k \geq 2^{128}$, this is sound and complete!

Multimodular Verification

[HofstadlerKaufmannChen IJCAR'26]



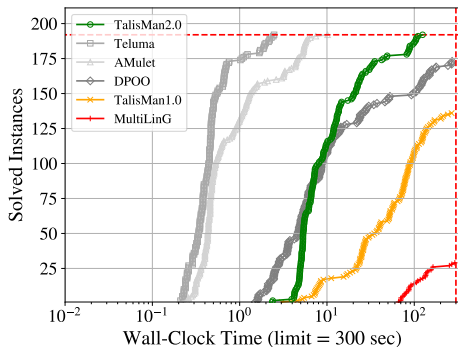
Theorem (Hofstadler, Kaufmann, Chen '26)

If $p_1 p_2 \cdots p_k \geq \mathbf{B}$, this is sound and complete!

Evaluation – Multiplier Circuits

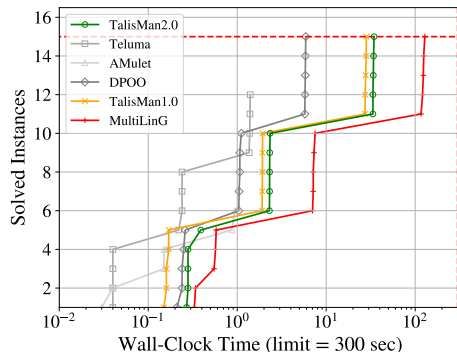
Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Optimized multipliers

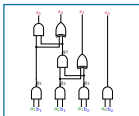
- 5 optimization scripts
- for 32, 64, 128 bit inputs



Is the circuit really really correct?

Proof Logging

Multiplier



Polynomials

$B = \{$
 $x - a_0 * b_0,$
 $y - a_1 * b_1,$
 $s_0 - x * y,$
 $\dots$
 $\}$

Specification

$$\sum_{i=0}^{2n-1} 2^i s_i =$$
$$\left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$

Ideal Membership

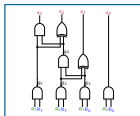
$= 0 \checkmark$
 $\neq 0 \times$

Correct?

Problem: Verification might not be error free

Proof Logging

Multiplier



Specification

$$\sum_{i=0}^{2n-1} 2^i s_i = \left(\sum_{i=0}^{n-1} 2^i a_i \right) \left(\sum_{i=0}^{n-1} 2^i b_i \right)$$

Polynomials

$B = \{$
 $x - a_0 * b_0,$
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 $\}$

Ideal Membership

$= 0 \checkmark$
 $\neq 0 \times$

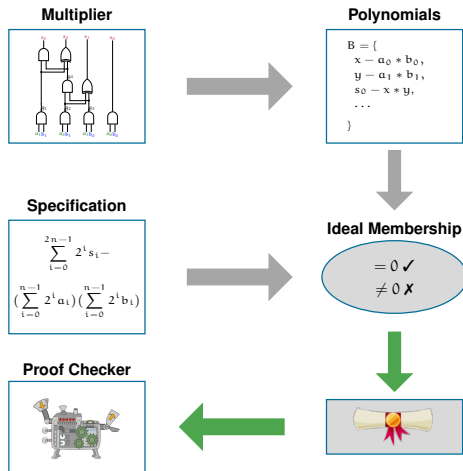


Problem: Verification might not be error free

Goal: Validate result of verification process

- Generate machine-checkable proofs
- Check by independent proof checkers

Proof Logging



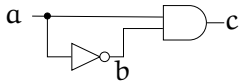
Problem: Verification might not be error free

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- Generate machine-checkable proofs
- Check by independent proof checkers

LPAC

Switch from explicit addition and multiplication rules to linear combinations.



$$P = 1 - b + 1 - a;$$

$$2 - c + a * b;$$

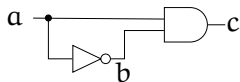
$$\text{Spec} = c$$

LPAC simulates PAC

```
3  % 1*(a), -a*b;  
1  d;  
4  % 3*(1)+2*(1), -c;  
2  d;  
3  d;  
5  % 4*(-1), c;
```

LPAC

Switch from explicit addition and multiplication rules to linear combinations.



$$P = 1 - b + 1 - a;$$

$$2 - c + a * b;$$

$$\text{Spec} = c$$

LPAC simulates PAC

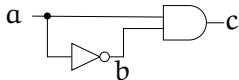
```
3  % 1*(a), -a*b;  
1  d;  
4  % 3*(1)+2*(1), -c;  
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LPAC

```
3  % 1*(a)+2*(1), -c;  
1  d;  
2  d;  
4  % 4*(-1), c;
```

LPAC simulates NSS

```
3  % 1*(-a)+2*(-1), -c;
```

D. Kaufmann, M. Fleury, A. Biere, M. Kauers, Practical Algebraic Calculus and Nullstellensatz with the Checkers Pacheck and Pastèque and Nuss-Checker FMDS, 2022.



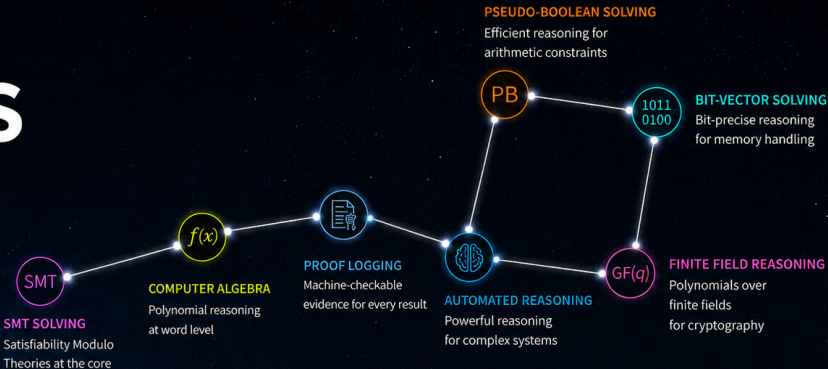
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Conclusion

Algebraic circuit verification (GB , Reduction )

Conclusion

Algebraic circuit verification (GB , Reduction )

 Flip things around: $\prec_{\text{lex}} \rightsquigarrow \prec_{\text{dlex}}$ (GB , Reduction )

Conclusion

Algebraic circuit verification (GB ✓, Reduction ⚠)

💡 Flip things around: $\prec_{\text{lex}} \rightsquigarrow \prec_{\text{dlex}}$ (GB ⚠, Reduction ✓)

Three approaches

Compute GB

- no requirements
- full GB computation
- double exponential runtime
- small circuits (≤ 10 variables)

FGLM

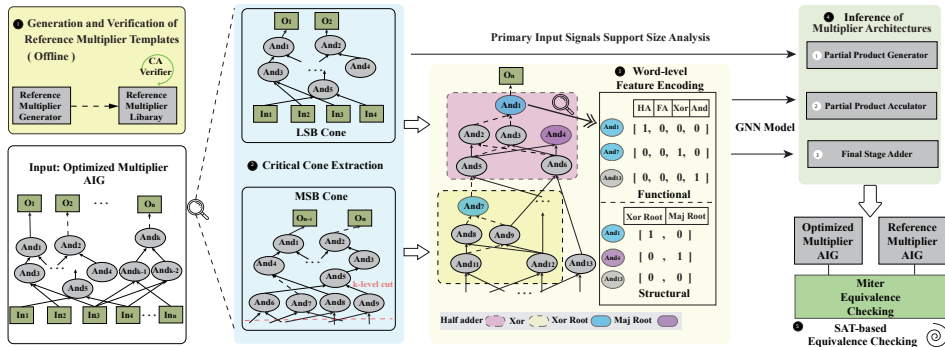
- requires a Gröbner basis
- first steps of FGLM
- exponential runtime
- moderate circuits (≤ 15 variables)

Guess & Prove

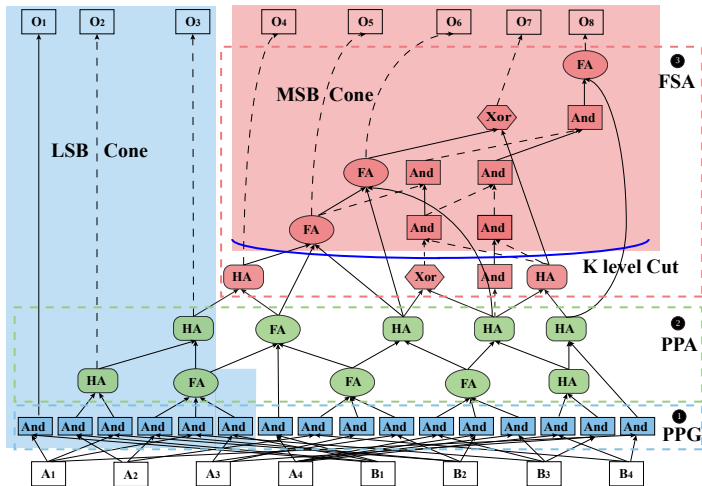
- requires variety
- interpolation (guess)
- uses SAT solver (prove)
- larger circuits (≥ 200 variables)

What about using GNNs?

ReVEAL - GNN-Guided Reverse Engineering for Formal Verification of Optimized Multipliers [ChenKaufmannDengSongZhangYu TACAS'26]



What about using GNNs?



What about using GNNs?

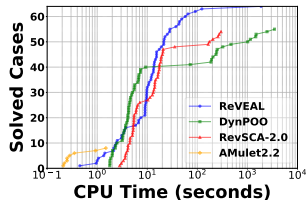
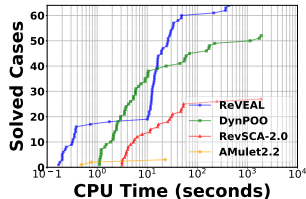


Table 3: Inference Accuracy for Each Stage

Stage	TOP 1*	TOP 2*	TOP 3*
Stage PPG	100%	—	—
Stage PPA	100%	—	—
Stage FSA	85%	97%	98%

* TOP 1, TOP 2, and TOP 3 represent the prediction accuracy when the ground truth is among the model's top 1, top 2, and top 3 ranked predictions, respectively.

Table 4: Solver Performance for Miter Solving

Opt	dc2			resyn3		
	Solved	Avg Time (s)	#. fastest	Solved	Avg Time (s)	#. fastest
cec	60/64	692.57	0	60/64	1007.20	1
&cec	64/64	8.50	46	64/64	6.83	50
&fraig -y	36/64	1848.53	12	39/64	1877.33	8
&fraig -x	37/64	1765.14	0	39/64	1726.71	5
kissat	64/64	96.79	6	64/64	186.55	0

References I

- [Biere SATComp'16] A. Biere. Collection of Combinational Arithmetic Miters Submitted to the SAT Competition 2016. In SAT Competition 2016, pages 65–66, Dep. of Computer Science Report Series B, University of Helsinki, 2016.
- [BiereFallerFazekasFleuryFroleyksPollitt SATComp'24] A. Biere, T. Faller, K. Fazekas, M. Fleury, N. Froleyks and F. Pollitt CaDiCaL, Gimsatul, IsaSAT and Kissat Entering the SAT Competition 2024. In SAT Competition 2024, pages 8–10, Dep. of Computer Science Report Series B, University of Helsinki, 2024.
- [Buchberger'65] B. Buchberger. Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal. PhD Thesis, University of Innsbruck, 1965.
- [CiesielskiYuBrownLiuRossi DAC'15] M. Ciesielski, C. Yu, W. Brown, D. Liu, and A. Rossi. Verification of Gate-level Arithmetic Circuits by Function Extraction. In Proc. of DAC'15, pages 52:1–52:6, ACM, 2015.
- [ChenBryant DAC'95] Y. Chen and R. Bryant. Verification of Arithmetic Circuits with Binary Moment Diagrams. In Proc. of DAC'95, pages 535–541, ACM, 1995.

References II

- [ChenKaufmannDengSongZhangYu TACAS'26] C. Chen, D. Kaufmann, C. Deng, Z. Song, H. Zhang, and C. Yu. ReVEAL: GNN-Guided Reverse Engineering for Formal Verification of Optimized Multipliers. In Proc. of TACAS'26, Springer, 2026.
- [DrechslerMahzoon ISEEIE'22] R. Drechsler and A. Mahzoon. Design Modification for Polynomial Formal Verification. In Proc. of ISEEIE'22, pages 187–194, IEEE, 2022.
- [KaufmannBeameBiereNordström DATE'22] D. Kaufmann, P. Beame, A. Biere and J. Nordström. Adding Dual Variables to Algebraic Reasoning for Gate-Level Multiplier Verification. In Proc. of DATE'22, pages 1431–1436, IEEE, 2022.
- [HofstadlerKaufmannChen IJCAR'26] C. Hofstadler, D. Kaufmann, and C. Chen. Avoiding Big Integers: Parallel Multimodular Algebraic Verification of Arithmetic Circuits. In Proc. of IJCAR'26.
- [KaufmannBiereKauers FMCAD'19] D. Kaufmann, A. Biere and M. Kauers. Verifying Large Multipliers by Combining SAT and Computer Algebra. In Proc. of FMCAD'19, pages 28–36, IEEE, 2019.
- [KaufmannBiereKauers FMSD'20] D. Kaufmann, A. Biere and M. Kauers. Incremental Column-Wise Verification of Arithmetic Circuits Using Computer Algebra. In FMSD, vol 56, pages 22–54, 2020.

References III

- [KaufmannFleuryBiere FMCAD'20] D. Kaufmann, M. Fleury and A. Biere. Pacheck and Pastèque Checking Practical Algebraic Calculus Proofs. In Proc. of FMCAD'20, pages 264–269, TU Vienna Academic Press, 2020.
- [KaufmannFleuryBiereKauers FMSD'21] D. Kaufmann, M. Fleury, A. Biere and M. Kauers. Practical Algebraic Calculus and Nullstellensatz with the Checkers Pacheck and Pastèque and Nuss-Checker. In FMSD, online first, 2021.
- [KonradScholl FMCAD'24] A. Konrad and C. Scholl. Practical Algebraic Calculus and Nullstellensatz with the Checkers Pacheck and Pastèque and Nuss-Checker. In FMSD, online first, 2021.
- [LvKallaEnescu TCAD'13] J. Lv, P. Kalla, F. Enescu. Efficient Gröbner Basis Reductions for Formal Verification of Galois Field Arithmetic Circuits. In IEEE TCAD, vol. 32, pages 1409–1420, 2013.
- [MahzoonGroßeDrechsler DAC'19] A. Mahzoon, D. Große and R. Drechsler. RevSCA: Using Reverse Engineering to Bring Light into Backwards Rewriting for Big and Dirty Multipliers. In Proc. of DAC'19, pages 185:1–185:6, ACM, 2019.

References IV

- [MahzoonGroßeDrechsler ICCAD'18] A. Mahzoon, D. Große and R. Drechsler. PolyCleaner: Clean your Polynomials before Backward Rewriting to verify Million-gate Multipliers. In Proc. of ICCAD'18, pages 129:1–129:8, ACM, 2018.
- [MahzoonGroßeSchollDrechsler DATE'20] A. Mahzoon, D. Große, Christoph Scholl and R. Drechsler. Towards Formal Verification of Optimized and Industrial Multipliers. In Proc. of DATE'20, pages 544–549, DATE, 2020.
- [RitircBiereKauers DATE'18] D. Ritirc, A. Biere and M. Kauers. Improving and Extending the Algebraic Approach for Verifying Gate-Level Multipliers. In Proc. of DATE'18, pages 1556–1561, IEEE, 2018.
- [RitircBiereKauers FMCAD'17] D. Ritirc, A. Biere and M. Kauers. Column-Wise Verification of Multipliers Using Computer Algebra. In Proc. of FMCAD'17, pages 23–30, IEEE, 2017.
- [SayedGroßeKühneSoekenDrechsler DATE'16] A. Sayed-Ahmed, D. Große, U. Kühne, M. Soeken, and R. Drechsler. Formal verification of integer multipliers by combining Gröbner basis with logic reduction. In Proc. of DATE'16, pages 1048–1053, IEEE, 2016.

References V

- [SchollKonradMahzoonGroßeDrechsler DATE'21] C. Scholl, A. Konrad, A. Mahzoon, D. Große and R. Drechsler. Verifying Dividers Using Symbolic Computer Algebra and Don't Care Optimization. In Proc. of DATE'21, pages 1110–1115, IEEE, 2021.
- [Temel TACAS'24] M. Temel eSCMul: Verified Implementation of S-C-Rewriting for Multiplier Verification. In Proc. of TACAS'24, pages 485–507, Springer, 2024.
- [TemelSlobodovaHunt CAV'20] M. Temel, A. Slobodova and W. Hunt. Automated and Scalable Verification of Integer Multipliers. In Proc. of CAV'20, pages 485–507, Springer, 2020.