SAT-GUIDED GRÖBNER BASIS METHODS FOR ARITHMETIC CIRCUIT VERIFICATION

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Invited Talk
SMT Workshop at FLoC, Lisbon, Portugal

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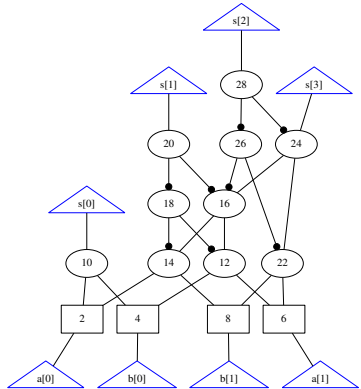
TU WIEN Informatics

FWF Austrian Science Fund

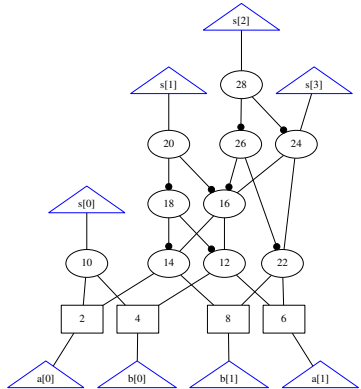
Circuit Verification



And-Inverter Graph

Circuit Verification

Is this a correct 2-bit multiplier?

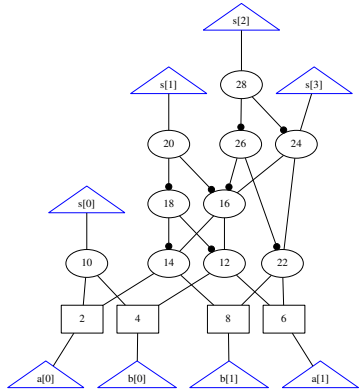


And-Inverter Graph

Circuit Verification

$$
\begin{array}{cccc}
1 & 1 & \cdot & 1 & 1 \\
\hline
 & & & 1 & 1 \\
 & & 1 & 1 & \\
 & 1 & 1 & 0 & \\
\hline
1 & 0 & 0 & 1 &
\end{array}
$$

$3 \cdot 3 = 9$



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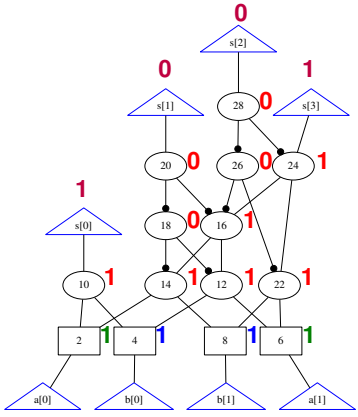


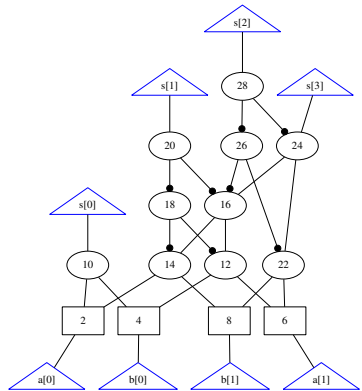
And-Inverter Graph

Circuit Verification

Given: Gate-level multiplier for fixed bit-width.

Question: For all possible $a_i, b_i \in \mathbb{B}$:

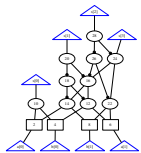
$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$



And-Inverter Graph

Formal Verification

System



Specification

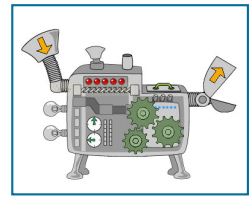
For all $a_i, b_i \in \mathbb{B}$:

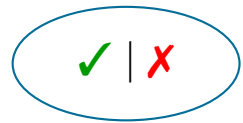
$$(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0?$$

Mathematical Model

$$\begin{aligned} B = \{ & \\ & x - a_0 * b_0, \\ & y - a_1 * b_1, \\ & s_0 - x * y, \\ & \dots \\ \} & \end{aligned}$$

Automated Decision Process





Formal Verification Techniques

Decision Diagrams

- First technique to detect Pentium bug

  [ChenBryant DAC'95]

- Requires knowledge of the layout

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Theorem Proving

- Used in industry, e.g., ACL2
  [TemelSlobodovaHunt CAV'20]
- Automated on the RTL level
  [Temel TACAS'24]

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Satisfiability Checking (SAT)

- SAT 2016: Exponential run-time of solvers [Biere SATComp'16]
- SAT 2024: Equivalence checking of structural similar circuits [BiereFallerFazekasFleuryFroleyksPollitt SATComp'24]

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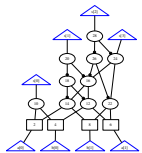
X^1 Algebraic Approach

- Seminal work: [LvKallaEnescu TCAD'13, CiesielskiYuBrownLiuRossi DAC'15] [SayedGroßeKühneSoekenDrechsler DATE'16]
- Polynomial encoding
- Works for non-trivial multiplier designs

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$$
\begin{aligned}
B = \{ & \\
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& s_0 - x * y, \\
& \ldots \\
\}
\end{aligned}
$$

Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right)$$

Implication

$= 0$ ✓

$\neq 0$ ✗

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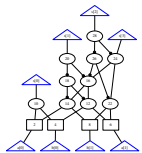

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Multiplier Specification

Unsigned Integers:

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right) \quad \in \mathbb{Q}[X]$$

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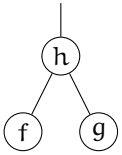
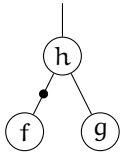
From AIGs to Polynomials



$h = f \wedge g$



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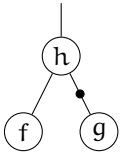

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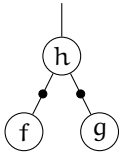

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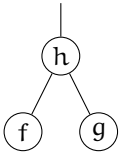
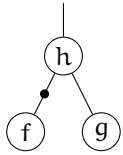


$h = f \wedge g$

f	g	h
0	0	0
0	1	0
1	0	0
1	1	1



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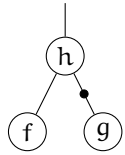

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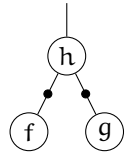

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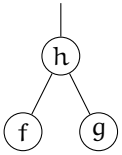

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0	0	0
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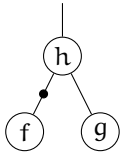
$-h + fg$



$h = \neg f \wedge g$

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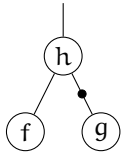
$-h - fg + g$



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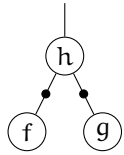

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0	0	1
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$-h + fg - f - g + 1$

From AIGs to Polynomials

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$
\begin{aligned}
&-s_3 + l_{24} \\
&-s_2 + l_{28} \\
&-s_1 + l_{20} \\
&-s_0 + l_{10} \\
&-l_{28} + l_{26}l_{24} - l_{26} - l_{24} + 1 \\
&-l_{26} + l_{22}l_{16} - l_{22} - l_{16} + 1 \\
&-l_{24} + l_{22}l_{16}
\end{aligned}
\qquad
\begin{aligned}
&-l_{22} + a_1 b_1 \\
&-l_{20} + l_{18}l_{16} - l_{18} - l_{16} + 1 \\
&-l_{18} + l_{14}l_{12} - l_{14} - l_{12} + 1 \\
&-l_{16} + l_{14}l_{12} \\
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&-l_{12} + a_1 b_0 \\
&-l_{10} + a_0 b_0
\end{aligned}
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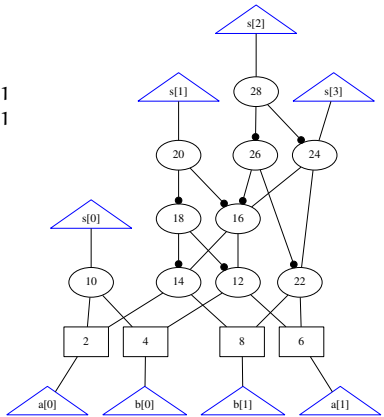

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-s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
-l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
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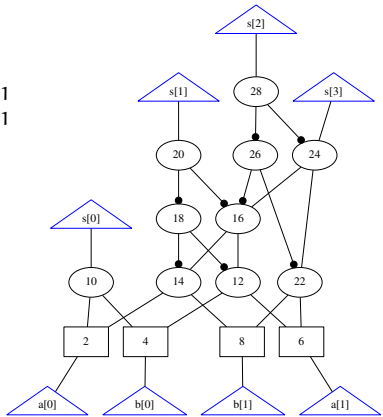
\end{array}
$$

Boolean input constraints $B(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$
\begin{array}{ll}
a_1, a_0 \in \mathbb{B} & -a_1^2 + a_1,\ -a_0^2 + a_0, \\
b_1, b_0 \in \mathbb{B} & -b_1^2 + b_1,\ -b_0^2 + b_0
\end{array}
$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1 a_1 - 2b_1 a_0 - 2b_0 a_1 - b_0 a_0$$



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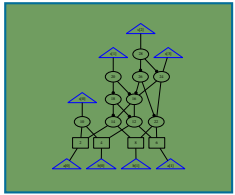

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Ideal Membership

$= 0$ ✓

$\neq 0$ ✗

COMPUTER ALGEBRA

COMPUTER ALGEBRA

IDEALS AND GRÖBNER BASES

Ideal

Definition: Ideal

A subset $I \subset \mathbb{K}[X]$ is an ideal if it satisfies:

- $0 \in I$
- If $f, g \in I$, then $f + g \in I$.
- If $f \in I$ and $h \in \mathbb{K}[X]$ then $hf \in I$.

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Ideal Generated by Finitely Many Polynomials

Let $f_1, \ldots, f_s \in \mathbb{K}[X]$. Then we set

$$\langle f_1, \ldots, f_s \rangle = \{h_1 f_1 + \cdots + h_s f_s \mid h_1, \ldots, h_s \in \mathbb{K}[X]\}.$$

$\langle f_1, \ldots, f_s \rangle$ is an ideal and is called the ideal generated by $f_1, \ldots, f_s$.

We can think of $\langle f_1, \ldots, f_s \rangle$ as consisting of all "polynomial consequences" of the equations $f_1 = f_2 = \ldots = f_s = 0$.

Applications of Ideals

The Ideal Membership Problem

Given $f \in \mathbb{K}[X]$ and an ideal $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[X]$, determine if $f \in I$.

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The Ideal Membership Problem

Given $f \in \mathbb{K}[X]$ and an ideal $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[X]$, determine if $f \in I$.

Idea Divide f by $f_1, \ldots, f_s$?

Ideal Membership Problem

Let $I = \langle x^2 - y,\ xy - x \rangle \subset \mathbb{Q}[x, y]$.
Is the polynomial $f = x^2 - y^2 \in I$?

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Is the polynomial $f = x^2 - y^2 \in I$?

Spoiler: Yes, because

$$y \cdot (x^2 - y) - x \cdot (xy - x) = x^2 - y^2$$

Ideal Membership Problem

Let $I = \langle x^2 - y,\ xy - x \rangle \subset \mathbb{Q}[x, y]$.
Is the polynomial $f = x^2 - y^2 \in I$?

$x^2 - y^2 \xrightarrow{x^2 - y} y - y^2$

$x^2 - y^2 \xrightarrow{xy - x} x^2 - y^2$ (no reduction possible)

Operation $\xrightarrow{p}$ is multivariate variant of polynomial division. Neither reduction yields 0.

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3. Every ideal $I \subseteq \mathbb{K}[X]$ has a Gröbner basis G w.r.t. a fixed monomial order $\prec$.
4. Given an arbitrary basis, we can compute G (double-exponential).

Orderings on the Monomials

Univariate Polynomials - Sort by Degree.

$$\cdots > x^{m+1} > x^m > \cdots > x^2 > x > 1$$

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$$x_1 > x_2 > \cdots > x_n$$

Question How to order non-linear multivariate polynomials in $\mathbb{K}[X]$?

Orderings on the Monomials

Let $\sigma_1 = x_1^{u_1} \cdots x_n^{u_n}$, $\sigma_2 = x_1^{v_1} \cdots x_n^{v_n}$.

Orderings on the Monomials

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Lexicographic Order $\prec_{\text{lex}}$

$\sigma_1 \prec_{\text{lex}} \sigma_2$ iff there exists an index i with $u_j = v_j$ for all $j < i$, and $u_i < v_i$.

Example: $x_1^2 x_2^7 \prec_{\text{lex}} x_1^3 x_2 \prec_{\text{lex}} x_1^3 x_2^6$

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Degree Lexicographic Order $\prec_{\text{dlex}}$

$\sigma_1 \prec_{\text{dlex}} \sigma_2$ iff either $|\sigma_1| < |\sigma_2|$ or $|\sigma_1| = |\sigma_2|$ and $\sigma_1 \prec_{\text{lex}} \sigma_2$.

Example: $x_1^3 x_2 \prec_{\text{dlex}} x_1^2 x_2^7 \prec_{\text{dlex}} x_1^3 x_2^6$

Computing a Gröbner Basis

Algorithm: Buchberger's Algorithm

Input : $F = \{f_1, \ldots, f_s\}$, monomial ordering $<$

Output: Gröbner basis $G = \{g_1, \ldots, g_t\}$ w.r.t. $<$, such that $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_t \rangle$

$G = F$;

$C = \{\{g_1, g_2\} \mid g_1, g_2 \in G, g_1 \neq g_2\}$;

while not all pairs $\{g_1, g_2\} \in C$ are marked **do**

  choose unmarked pair $\{g_1, g_2\}$;

  mark $\{g_1, g_2\}$;

  $h = \text{normalform of } \mathrm{spol}(g_1, g_2) \text{ w.r.t. } G \qquad (\mathrm{spol}(g_1, g_2) \xrightarrow{G} h)$;

  if $h \neq 0$ **then**

    $C = C \cup \{\{g, h\} \mid g \in G\}$;

    $G = G \cup \{h\}$;

return G

S-polynomial. For $p, q \in \mathbb{K}[X]$ with $L = \mathrm{lcm}(\mathrm{lm}(p), \mathrm{lm}(q))$: $\mathrm{spol}(p, q) = \frac{L}{\mathrm{lt}(p)}\, p - \frac{L}{\mathrm{lt}(q)}\, q$

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Product Criterion. If $p, q \in k[x_1, \ldots, x_n] \setminus \{0\}$ are such that the leading monomials are coprime, i.e., $\mathrm{lcm}(\mathrm{lm}(p), \mathrm{lm}(q)) = \mathrm{lm}(p)\,\mathrm{lm}(q)$, then $\mathrm{spol}(p, q)$ reduces to zero mod $\{p, q\}$.

Computing a Gröbner Basis - Example

Recall $I = \langle x^2 - y,\ xy - x \rangle \subset \mathbb{Q}[x, y]$, order $\prec_{\text{lex}}$ with $x > y$.
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Reduce w.r.t. G: $x^2 - y^2 \xrightarrow{f_1} y - y^2 \neq 0$

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Remaining pairs. $\operatorname{spol}(f_1, f_3) \xrightarrow{G} 0, \qquad \operatorname{spol}(f_2, f_3) \xrightarrow{G} 0$

$\Rightarrow G = \{x^2 - y,\ xy - x,\ y^2 - y\}$ is a Gröbner basis of I.

Ideal Membership - Revisited

Recall $f = x^2 - y^2$ and the Gröbner basis $G = \{x^2 - y,\ xy - x,\ y^2 - y\}$.

$$x^2 - y^2 \xrightarrow{x^2 - y} y - y^2 \xrightarrow{y^2 - y} 0$$

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$$x^2 - y^2 \xrightarrow{x^2 - y} y - y^2 \xrightarrow{y^2 - y} 0$$

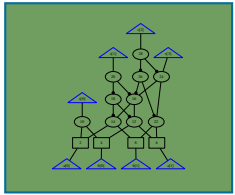
The remainder of f w.r.t. the Gröbner basis G is 0.

$\Rightarrow f \in I$

BACK TO CIRCUITS

Basic Idea of Algebraic Approach

Multiplier



Polynomials

$$
\begin{aligned}
B = \{ & \\
& x - a_0 * b_0, \\
& y - a_1 * b_1, \\
& s_0 - x * y, \\
& \ldots \\
\} &
\end{aligned}
$$

Specification

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right)$$

Ideal Membership

$= 0$ ✓

$\neq 0$ ✗

Ideal Membership Problem

- Polynomial Encoding:
  - Gate polynomials $\mathrm{G}(\mathrm{C})$
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Ideal Membership Problem

- Polynomial Encoding:
  - Gate polynomials $G(C)$
  - Boolean input constraints $B(C)$
- Let $I(C) = \langle G(C) \cup B(C) \rangle$.

Definition: Multiplier

A circuit C is called a multiplier if

$$\sum_{i=0}^{2n-1} 2^i s_i - \left(\sum_{i=0}^{n-1} 2^i a_i\right)\left(\sum_{i=0}^{n-1} 2^i b_i\right) \quad \in \quad I(C).$$

Verification Algorithm

Theorem (Kaufmann, Biere, Kauers FMCAD'17) For any circuit C, the set $G(C) \cup B(C)$ forms a Gröbner basis for $I(C)$ w.r.t. any lexicographic monomial order where "gate inputs $<$ gate output".

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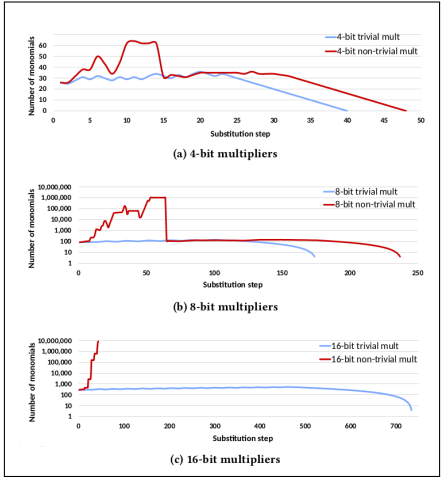
Computational Problems

- 👎 The number of monomials in the intermediate results blows-up.
- 👎 8-bit multiplier cannot be verified within 20 minutes.

Verification Algorithm

[MahzoonGroßeDrechsler ICCAD'18]



Strategies

- Encoding
  - Embedding negations as literals [KaufmannBeameBiereNordström DATE'22, KonradScholl FMCAD'24]
- Preprocessing
  - Variable Elimination [MahzoonGroßeDrechsler DAC'19, RitircBiereKauers DATE'18]
- Reduction
  - Incremental Algorithm [RitircBiereKauers FMCAD'17]
  - Dynamic Reduction Order [MahzoonGroßeSchollDrechsler DATE'20, KonradScholl FMCAD'24]
- Tricky: OR Gates in final stage adder
  - Include SAT or BDDs [KaufmannBiereKauers FMCAD'19, DrechslerMahzoon ISEEIE'22]

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Demo: `d-kfmnn.github.io/circa`

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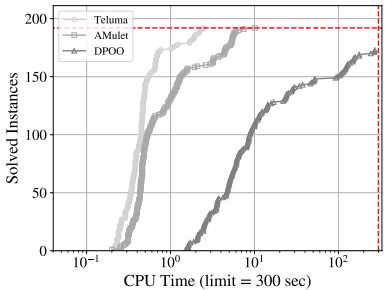
All of these strategies rely on a **lexicographic term ordering** and aim to find an optimal reduction order.

Strategies

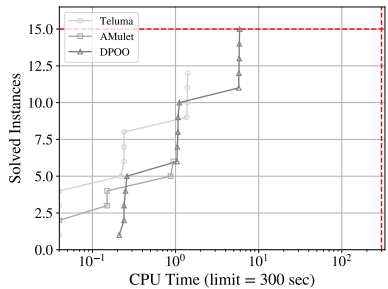
Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Optimized multipliers

- 5 optimization scripts
- for 32, 64, 128 bit inputs



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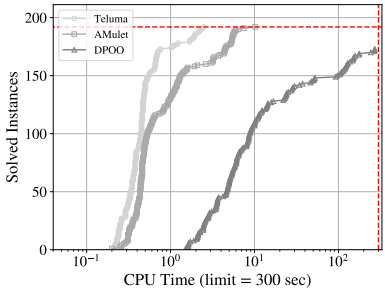

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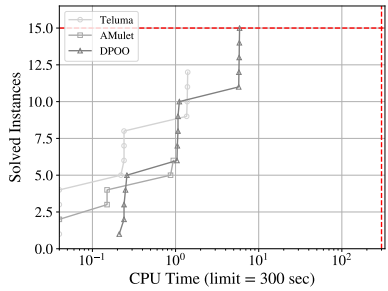

👎 Fast approaches are overfitted

A New Idea

- Change monomial order to degree-lexicographic order (Kaufmann, Berthomieu TACAS'25)

	$\prec_{\text{lex}}$	$\prec_{\text{dlex}}$
GB Computation	✅ Easy	⚠️ Hard
Spec Reduction	⚠️ Hard	✅ Easy

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	$\prec_{\text{lex}}$	$\prec_{\text{dlex}}$
GB Computation	Easy	Hard
Spec Reduction	Hard	Easy

Theorem (Kaufmann, Berthomieu TACAS'25) If the specification polynomial is linear, then the linear polynomials in a $\prec_{\text{dlex}}$-GB suffice to verify the circuit.

Verification via Linearization

Input: Circuit C, specification polynomial S
Output: Determine whether C fulfils S

1 $G \leftarrow G(C) \cup B(C)$
2 $S_{\text{lin}}, G \leftarrow$ Linearize-Spec(S, G)
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5 **return** $S_{\text{lin}} \stackrel{?}{=} 0$

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$S_{\text{lin}}, G_{\text{ext}} \leftarrow$ **Linearize-Spec**(S, G)

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$
\begin{aligned}
&-s_3 + l_{24} && -l_{22} + a_1 b_1 \\
&-s_2 + l_{28} && -l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1 \\
&-s_1 + l_{20} && -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\
&-s_0 + l_{10} && -l_{16} + l_{14} l_{12} \\
&-l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 && -l_{14} + a_0 b_1 \\
&-l_{26} + l_{22} l_{16} - l_{22} - l_{16} + 1 && -l_{12} + a_1 b_0 \\
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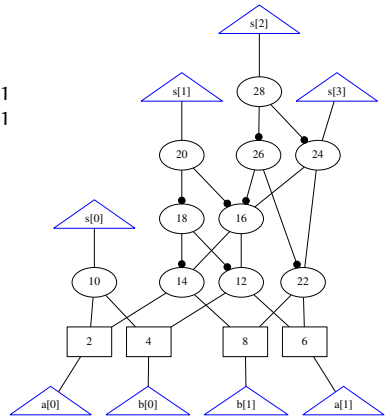
$$



Specification $S \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4b_1 a_1 - 2b_1 a_0 - 2b_0 a_1 - b_0 a_0$$

$S_{lin}, G_{ext} \leftarrow$ **Linearize-Spec**(S, G)

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Extension polynomials $G_{ext} \subseteq \mathbb{Z}^{2n}[X]$.

$$-t_{11} + a_1 b_1 \qquad -t_{01} + a_0 b_1$$
$$-t_{10} + a_1 b_0 \qquad -t_{00} + a_0 b_0$$

Linear Specification $S_{lin} \in \mathbb{Z}^{2n}[X]$.

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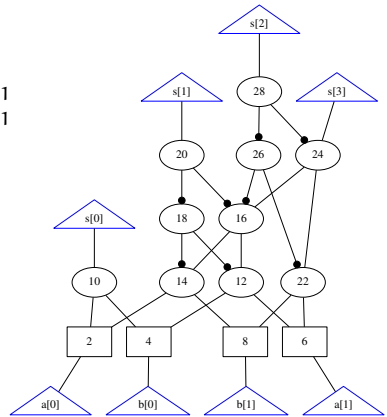

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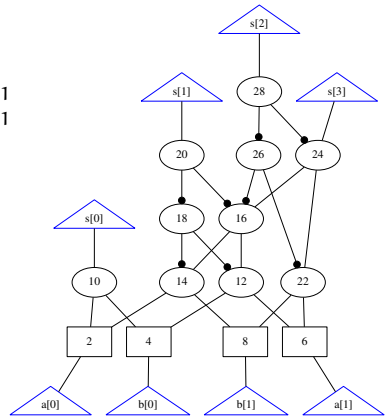

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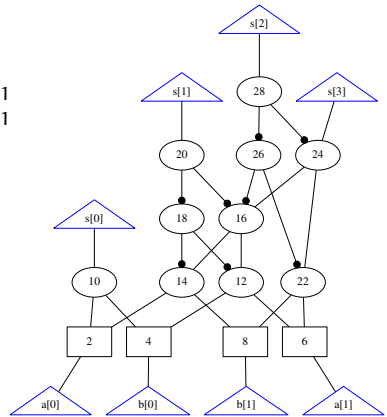

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compute $\prec_{\text{dlex}}$-GB of G

Computing a $\prec_{\text{dlex}}$-GB for the whole circuit is infeasible.

Verification via Linearization

Input: Circuit C, specification polynomial S
Output: Determine whether C fulfils S

1 $G \leftarrow G(C) \cup B(C)$
2 $S_{\text{lin}}, G \leftarrow$ Linearize-Spec(S, G)
3 $L \leftarrow$ LINEAR-POLIES-OF-$\prec_{\text{dlex}}$-GB(G)
4 $S_{\text{lin}} \leftarrow$ Linear-Reduce(S_{lin}, L)
5 **return** $S_{\text{lin}} \stackrel{?}{=} 0$

compute $\prec_{\text{dlex}}$-GB of $G' \subseteq G$

Computing a $\prec_{\text{dlex}}$-GB for the whole circuit is infeasible.

$\rightsquigarrow$ work locally: extract sub-circuits and only compute $\prec_{\text{dlex}}$-GBs for them

Local Linearization

Gate polynomials $G(C) \subseteq \mathbb{Z}^{2n}[X]$.

$$
\begin{array}{ll}
-s_3 + l_{24} & -l_{22} + a_1 b_1 \\
-s_2 + l_{28} & -l_{20} + l_{18} l_{16} - l_{18} - l_{16} + 1 \\
-s_1 + l_{20} & -l_{18} + l_{14} l_{12} - l_{14} - l_{12} + 1 \\
-s_0 + l_{10} & -l_{16} + l_{14} l_{12} \\
-l_{28} + l_{26} l_{24} - l_{26} - l_{24} + 1 & -l_{14} + a_0 b_1 \\
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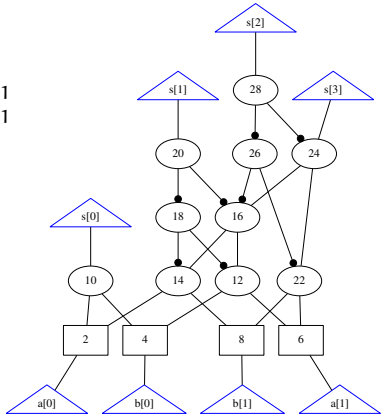
-l_{24} + l_{22} l_{16} & -l_{10} + a_0 b_0
\end{array}
$$

Specification $\mathcal{S}_n \in \mathbb{Z}^{2n}[X]$.

$$8s_3 + 4s_2 + 2s_1 + s_0 - 4l_{22} - 2l_{14} - 2l_{12} - l_{10}$$



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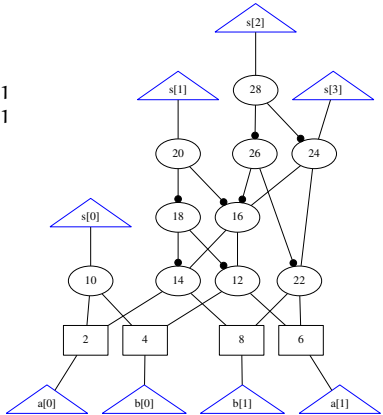

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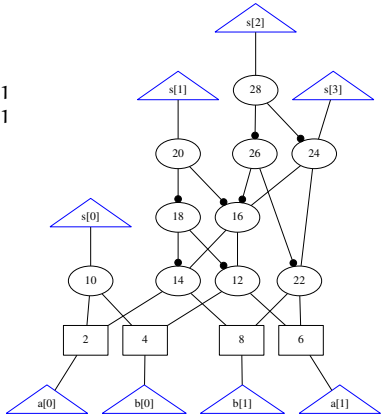

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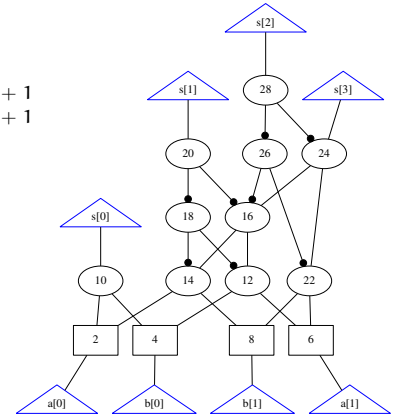

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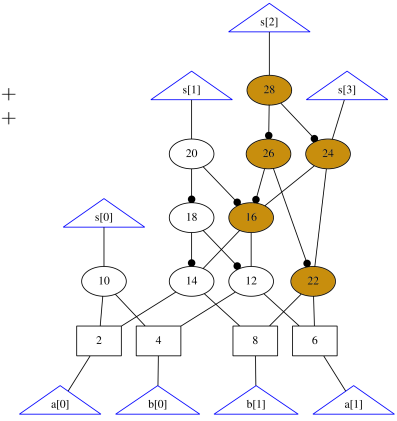

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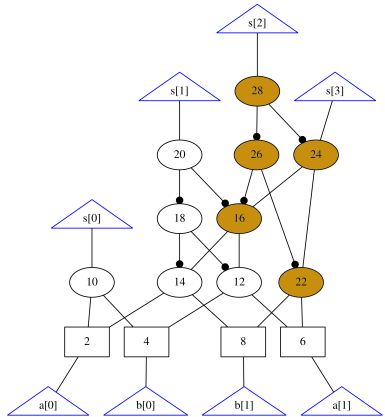

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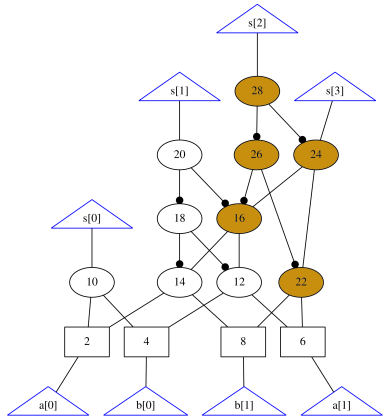
$$



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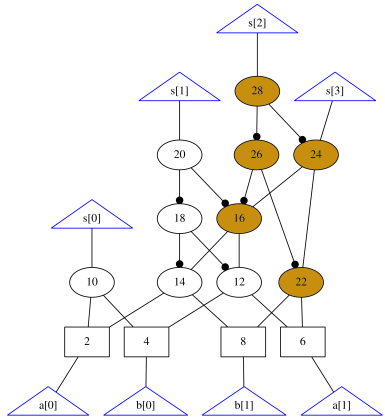

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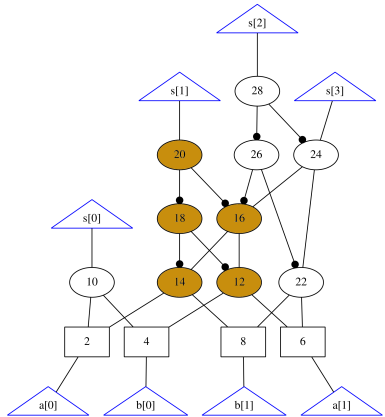

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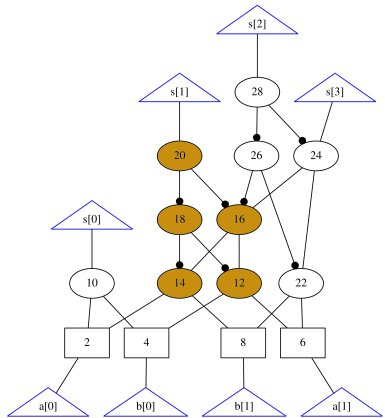
- 2l_{18} + 2l_{16} - 2l_{14} - 2l_{12} + 2
\end{aligned}
$$



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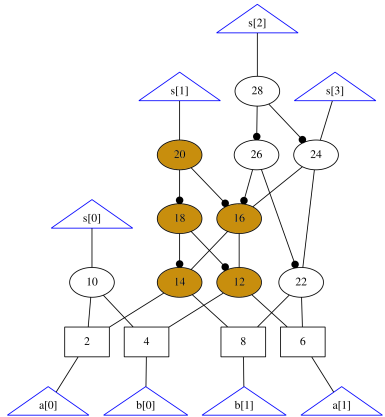
&-2l_{18} + 2l_{16} - 2l_{14} - 2l_{12} + 2 \\
&0
\end{aligned}
$$

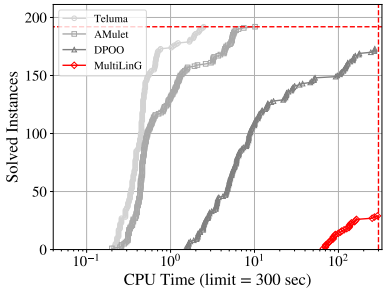


First Approach

(Kaufmann, Berthomieu TACAS'25)

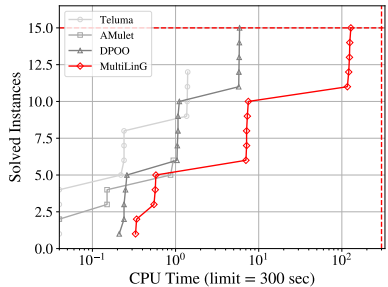
Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Optimized multipliers

- 5 optimizatoin scripts
- for 32, 64, 128 bit inputs



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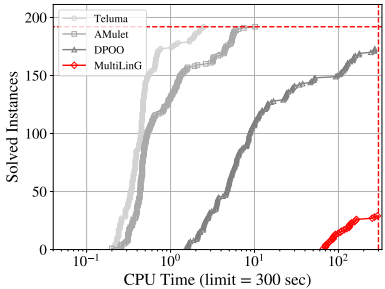

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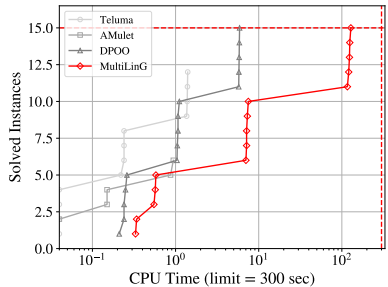

- Approach seems to be more robust against optimization
- Black-box GB computation is too costly (can handle ≤ 8 vars)
- Completely ignores existing structure

Two Targeted Approaches

(Hofstadler, Kaufmann, CP'25)

FGLM-style Linearization

Guess & Prove Linearization

Two Targeted Approaches

(Hofstadler, Kaufmann, CP'25)

FGLM-style Linearization

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FGLM-style Linearization

- Exploit that we already have ($\prec_{\mathrm{lex}}$-)Gröbner basis.

FGLM-style Linearization

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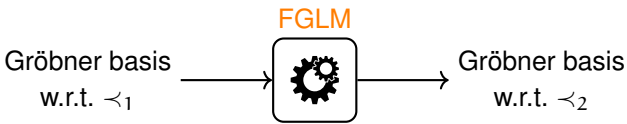
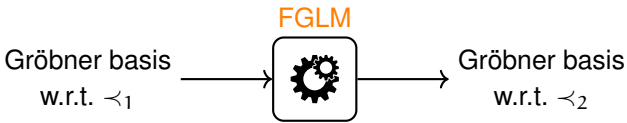


One can adapt FGLM to compute all linear polynomials and then stop.

Exploit existing structure $\rightsquigarrow$ a lot faster than GB computation

FGLM-style Linearization

- Exploit that we already have ($\prec_{\text{lex}}$-)Gröbner basis.



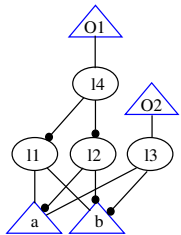
One can adapt FGLM to compute all linear polynomials and then stop.

- Exploit existing structure $\rightsquigarrow$ a lot faster than GB computation
- Runtime exponential in number of variables (can handle ≤ 15 vars)

FGLM-style Linearization

Ideal $I \subseteq \mathbb{Q}[a, b, l_1, l_2, l_3, l_4]$ **generated by:**

$$l_1 - ab, \quad l_2 - (1-a)(1-b), \quad l_3 - a(1-b), \quad l_4 - (1-l_1)(1-l_2), \quad a^2 - a, \quad b^2 - b$$



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1. Compute Normalforms:

$$l_1 = ab, \quad l_2 = ab - a - b + 1, \quad l_3 = -ab + a, \quad l_4 = -2ab + a + b$$

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$$l_1 = ab, \quad l_2 = ab - a - b + 1, \quad l_3 = -ab + a, \quad l_4 = -2ab + a + b$$

2. Build Coefficient Matrix A:

$$\begin{matrix} \mathrm{NF}_G(1) & \mathrm{NF}_G(a) & \mathrm{NF}_G(b) & \mathrm{NF}_G(l_1) & \mathrm{NF}_G(l_2) & \mathrm{NF}_G(l_3) & \mathrm{NF}_G(l_4) \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -2 \end{pmatrix} \begin{matrix} 1 \\ a \\ b \\ ab \end{matrix}$$

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3. Compute Kernel of A:

$$\begin{array}{c} \begin{array}{ccccccc} 1 & a & b & l_1 & l_2 & l_3 & l_4 \end{array} \\ \begin{pmatrix} 0 & -1 & -1 & 2 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

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4. Read off Linear Polynomials:

$$l_4 + 2l_1 - a - b, \qquad l_3 + l_1 - a, \qquad l_2 - l_1 + a + b - 1$$

Two Targeted Approaches

(Hofstadler, Kaufmann, CP'25)

FGLM-style Linearization

Guess & Prove Linearization

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Guess & Prove Linearization

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- Exploit that we can compute (a lot of) points in the variety $\mathrm{V}(\mathrm{G})$

Guess & Prove Linearization

- Exploit that we can compute (a lot of) points in the variety $V(G)$

  - ✓ our ideals are radical
  - ✓ our ideals are zero-dimensional
  - ✓ our ideals don't have additional roots in algebraic closure
  - ✓ we can efficiently test ideal membership of linear polynomials

Guess & Prove Linearization

Given: $G \subseteq \mathbb{K}[X]$ satisfying the assumptions from before

Compute: $\mathbb{K}$-VS basis of $\{ f \in \langle G \rangle \mid \deg(f) \leq 1 \}$

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Algorithm

1. Sample points $P \subseteq V(G)$
2. Compute all linear polynomials F that vanish on P
3. **if** $F \subseteq \langle G \rangle$
4. **return** F
5. **else** add witness points to P and repeat

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4. **return** F "repair"
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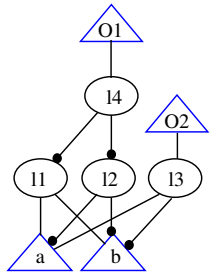

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Matrix A**:**

$$\begin{array}{c} \begin{matrix} 1 & a & b & l_1 & l_2 & l_3 & l_4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

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Kernel yields Guesses:

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In this case, SAT solver returns SAT and $p = (1, 1, 1, 0, 0, 0)$.

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Works really well also for larger circuits (can handle ≥ 200 variables)!

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...how many iterations of the repair loop do we need? ≤ 3

Verification via Linearization

Input: Circuit C, specification polynomial S
Output: Determine whether C fulfils S

1 $G \leftarrow G(C) \cup B(C)$
2 $S_{\text{lin}}, G \leftarrow$ Linearize-Spec(S, G)
3 $L \leftarrow$ LINEAR-POLIES-OF-$\prec_{\text{dlex}}$-GB(G)
4 $S_{\text{lin}} \leftarrow$ Linear-Reduce(S_{lin}, L)
5 **return** $S_{\text{lin}} \stackrel{?}{=} 0$

compute $\prec_{\text{dlex}}$-GB of $G' \subseteq G$

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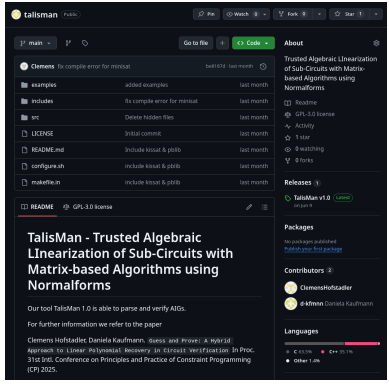
pick $G' \subseteq G$
if $|G'|$ is small use FGLM
else use Guess & Prove

TALISMAN

https://github.com/d-kfmnn/talisman/

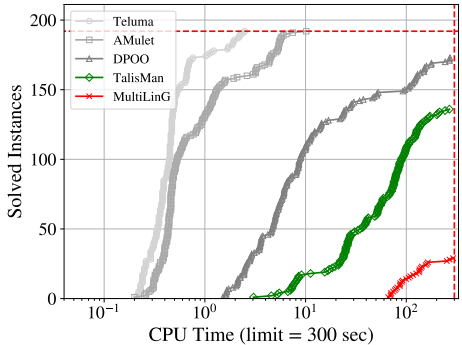
- Uses FLINT for matrix computations
- Guess and Prove Algorithm:
  - Uses PBLIB to translate guessed polynomial to CNF
  - Uses KISSAT as a SAT solver
  - Uses caching of isomorphic subcircuits



Evaluation – Multiplier Circuits

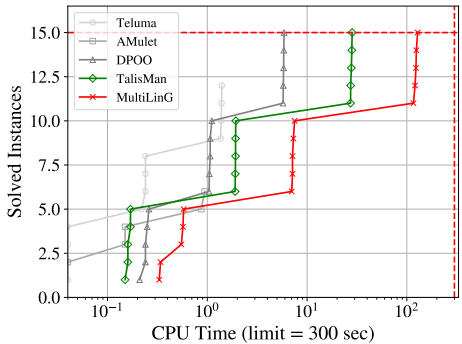
Unoptimized multipliers

- 192 architectures
- 64 bit inputs



Optimized multipliers

- 5 optimization scripts
- for 32, 64, 128 bit inputs



Teaser – TALISMAN 2 at IJCAR26

$$S \rightsquigarrow \cdots \rightsquigarrow \sum_{i=1}^{D} c_i \cdot m_i \rightsquigarrow \cdots \rightsquigarrow 0$$

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Number of terms

D

Can grow exponentially

Extensive prior work:

Encodings, Preprocessing, Dynamic division order, SAT/BDD-reasoning

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Coefficients

c_i

Values up to $\geq 2^{127}$

👎 Arbitrary precision arithmetic
👎 CNF Translation explodes

Largely ignored so far!

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Talk on Wednesday

Conclusion

Algebraic circuit verification (GB ✅, Reduction ⚠️)

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💡 Flip things around: $\prec_{\text{lex}} \rightsquigarrow \prec_{\text{dlex}}$ (GB ⚠️, Reduction ✅)

Conclusion

Algebraic circuit verification (GB ✅, Reduction ⚠️)

💡 Flip things around: $\prec_{\text{lex}} \rightsquigarrow \prec_{\text{dlex}}$ (GB ⚠️, Reduction ✅)

Three approaches

Compute GB

- no requirements
- full GB computation
- double exponential runtime
- small circuits (≤ 10 variables)

FGLM

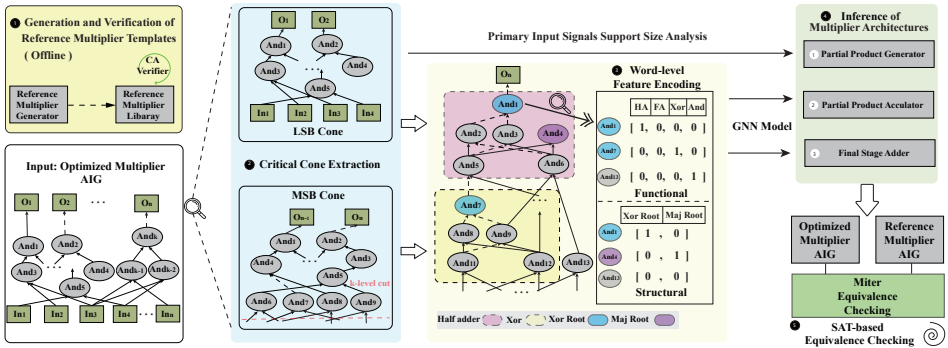
- requires a Gröbner basis
- first steps of FGLM
- exponential runtime
- moderate circuits (≤ 15 variables)

Guess & Prove

- requires variety
- interpolation (guess)
- uses SAT solver (prove)
- larger circuits (≥ 200 variables)

What about using GNNs?

ReVEAL - GNN-Guided Reverse Engineering for Formal Verification of Optimized Multipliers [ChenKaufmannDengSongZhangYu TACAS'26]



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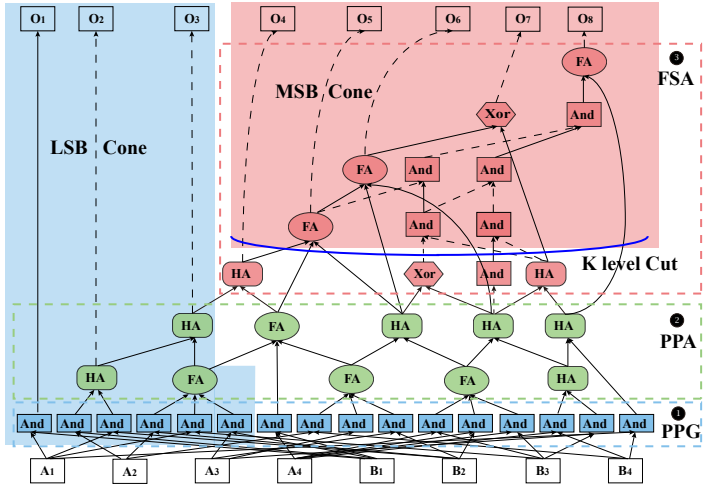

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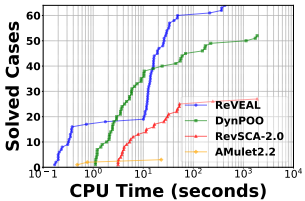

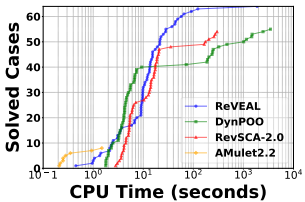

Table 3: Inference Accuracy for Each Stage

Stage	TOP 1*	TOP 2*	TOP 3*
Stage PPG	100%	–	–
Stage PPA	100%	–	–
Stage FSA	85%	97%	98%

* TOP 1, TOP 2, and TOP 3 represent the prediction accuracy when the ground truth is among the model's top 1, top 2, and top 3 ranked predictions, respectively.

Table 4: Solver Performance for Miter Solving

Opt	dc2			resyn3		
	Solved	Avg Time (s)	#. fastest	Solved	Avg Time (s)	#. fastest
cec	60/64	692.57	0	60/64	1007.20	1
&cec	64/64	8.50	46	64/64	6.83	50
&fraig -y	36/64	1848.53	12	39/64	1877.33	8
&fraig -x	37/64	1765.14	0	39/64	1726.71	5
kissat	64/64	96.79	6	64/64	186.55	0

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