A PRACTICAL POLYNOMIAL CALCULUS FOR ARITHMETIC CIRCUIT VERIFICATION

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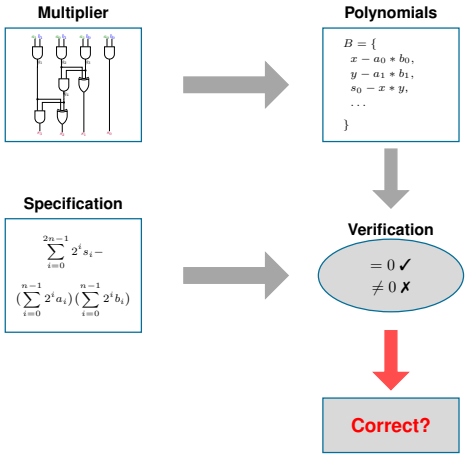
RiSE
Rigorous Systems Engineering

Motivation



Problem: Verification might not be error free

Goal: Validate result of verification process

- Generate machine-checkable proofs
- Check by independent proof checkers

Contribution:

- Proof format based on polynomial calculus
- Independent proof checker
- Validate previous verification results

Related Work

- **Reasoning with polynomial equations**
  - D. Kapur. Using **Gröbner Bases** to **Reason** About Geometry Problems. In JSC, 1986.
- **Algebraic proof systems**
  - M. Clegg, J. Edmonds and R. Impagliazzo. Using the **Groebner basis algorithm** to find proofs of unsatisfiability. In STOC, 1996.
  - P. Beame, R. Impagliazzo, J. Krajícek, T. Pitassi and P. Pudlák. Lower bounds on **Hilbert's Nullstellensatz** and propositional proofs. In Proc. London Math. Soc., 1996.
- **Proof complexity**
  - R. Impagliazzo, P. Pudlák and J. Sgall. **Lower bounds** for the polynomial calculus and the Gröbner basis algorithm. In Computational Complexity, 1999.
  - M. Lauria and J. Nordström. **Graph Colouring** is Hard for Algorithms Based on Hilbert's Nullstellensatz and Gröbner Bases. In LIPIcs, 2017.

Preliminaries

Let $f \in \mathbb{F}[X]$.

Polynomial function. $X \mapsto f(X)$

Polynomial equation. $f(X) = 0$ solutions are the roots of f

Ideal. A nonempty subset $I \subseteq \mathbb{F}[X]$ is called an ideal if

$$\forall\, p, q \in I : p + q \in I \qquad \text{and} \qquad \forall\, p \in \mathbb{F}[X]\ \forall\, q \in I : pq \in I$$

Basis. A set $P = \{p_1, \ldots, p_m\} \subseteq \mathbb{F}[X]$ is called a **basis** of an ideal I if

$$I = \{q_1 p_1 + \cdots + q_m p_m \mid q_1, \ldots, q_m \in \mathbb{F}[X]\} = \langle P \rangle$$

Polynomial Calculus

Let $G \subseteq \mathbb{F}[X]$ and $f \in \mathbb{F}[X]$.

Proof: Sequence $P = (p_1, \dots p_n)$, where each p_k is obtained by one of the rules:

Addition	$\dfrac{p_i \qquad p_j}{p_i + p_j}$	p_i, p_j appearing earlier in the proof or are contained in G
Multiplication	$\dfrac{p_i}{qp_i}$	p_i appearing earlier in the proof or is contained in G and $q \in \mathbb{F}[X]$ being arbitrary

If $p_n = f$ we write $G \vdash f$ or in algebraic terms $f \in \langle G \rangle$.

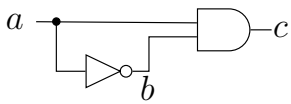
Refutation: $G \vdash 1$ or $1 \in \langle G \rangle$.

Gröbner Basis

Because of Gröbner bases the polynomial calculus is decidable:

- Buchberger's algorithm computes a Gröbner basis $G = \{g_1, \ldots, g_m\}$ for the ideal $\langle H \rangle$ generated by H.
- Given a Gröbner basis G, there is a computable function $\mathrm{red}_G \colon \mathbb{F}[X] \to \mathbb{F}[X]$ such that $\forall\, f \in \mathbb{F}[X] : \mathrm{red}_G(f) = 0 \iff f \in \langle G \rangle$.

Example



$$
\begin{aligned}
G &= \{ \quad -b+1-a, && b=\neg a \\
&\qquad -c+ab, && c=a\wedge b=a\wedge\neg a \\
&\qquad a^2-a,\ b^2-b,\ c^2-c\} && a,b,c\in\mathbb{B} \\
f &= c
\end{aligned}
$$

$$\mathrm{red}_G(f) = 0$$

$$
\begin{array}{lcl}
 & * & \dfrac{-b+1-a}{-ab+a-a^2} \\
+ & & \dfrac{a^2-a}{\quad} \\
 & + & \dfrac{-ab}{\quad} \qquad \dfrac{-c+ab}{\quad} \\
 & & * \ \dfrac{-c}{c}
\end{array}
$$

$$P = (-ab+a-a^2,\ -ab,\ -c,\ c)$$

Extended Calculus

Radical $\dfrac{p_i^m}{p_i}$ $m \in \mathbb{N} \setminus \{0\}$ and p_i^m appearing earlier in the proof or is contained in G.

If f can be deduced from polynomials $g \in G$, we write $G \vdash^+ f$ or $f \in \sqrt{\langle G \rangle}$.

Radical ideal. $\sqrt{\langle G \rangle} = \{f \in \mathbb{F}[X] \mid G \vdash^+ f\} = \{f \in \mathbb{F}[X] \mid f^m \in \langle G \rangle \text{ for } m \in \mathbb{N} \setminus \{0\}\}$

The extended calculus $\vdash^+$ is decidable. It can be reduced to $\vdash$ with the Rabinowitsch trick:

$$f \in \sqrt{\langle G \rangle} \iff 1 \in \langle G \cup \{yf - 1\} \rangle \quad \text{or} \quad G \vdash^+ f \iff G \cup \{yf - 1\} \vdash 1$$

Models

Let $G \subseteq \mathbb{F}[X]$ and $f \in \mathbb{F}[X]$.

- A **model** for G is a point $u = (u_1, \ldots, u_n) \in \mathbb{F}^n$ such that for all $g \in G$ it holds that $g(u_1, \ldots, u_n) = 0$.
- We write $G \models f$ if every model for G is also a model for $\{f\}$.
- For an algebraically closed field $\mathbb{F}$ we can deduce with Hilbert's Nullstellensatz that $G \models f \iff G \vdash^+ f$.
- PC including the radical rule is correct ("$\Leftarrow$") and complete ("$\Rightarrow$").
- In combination with Rabinowitsch's trick, we can decide the existence of models and produce certificates for the non-existence of models.

Models in $\{0,1\}^n$

For our applications, only models $u \in \{0,1\}^n \subseteq \mathbb{F}^n$ matter.

- Let us write $G \models_{\mathsf{bool}} f$ if every model $u \in \{0,1\}^n$ of G is also a model of $\{f\}$.
- We have $G \models_{\mathsf{bool}} f \iff G \cup B \models f$, where $B = \{x_i(x_i - 1) : i = 1, \ldots, n\}$.
- $G \cup B \models f \iff G \cup B \vdash^+ f$ holds also when $\mathbb{F}$ is not algebraically closed.
- The finiteness of $\{0,1\}^n$ also implies that $G \cup B \vdash^+ f \iff G \cup B \vdash f$ (Seidenberg's lemma).

Practical Algebraic Calculus

We translate the polynomial calculus into a more concrete proof format:

- For correctness it is important to know how the polynomials in the proof where derived
- Usually known $\rightarrow$ store this information

Practical Algebraic Calculus (PAC)

allows automated proof checking

PAC Syntax

letter ::= ‘a’ | ‘b’ | ... | ‘z’ | ‘A’ | ‘B’ | ... | ‘Z’

number ::= ‘0’ | ‘1’ | ... | ‘9’

constant ::= (number)$^{+}$

variable ::= letter (letter | number)*

power ::= variable [‘^’ constant]

term ::= power (‘*’ power)*

monomial ::= constant | [constant ‘*’] term

operator ::= ‘+’ | ‘-’

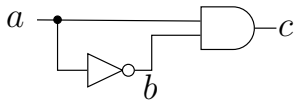
polynomial ::= [‘-’] monomial (operator monomial)*

given ::= (polynomial ‘;’)*

rule ::= (‘+’ | ‘*’) ‘:’ polynomial ‘,’ polynomial ‘,’ polynomial ‘;’

proof ::= (rule ‘;’)*

Example - PAC



$$G = \{ \; -b+1-a, \; -c+ab, \; a^2-a, \; b^2-b, \; c^2-c \}$$

$$f = c$$

```
* : -b+1-a,        a,          -a*b+a-a^2;
+ : -a*b+a-a^2,    a^2-a,      -a*b;
+ : -a*b,          -c+a*b,     -c;
* : -c,            -1,         c;
```

Proof Checking

A proof **rule** contains four components:

$$o : v, w, \ p;$$

Proof checking:

- Connection property: v, w are given polynomials or conclusions p_i of previous rules
- Inference property: verify correctness of each rule, e.g. $p = v + w$ for $o = " + "$
- Refutation check: at least one p_i is a non-zero constant

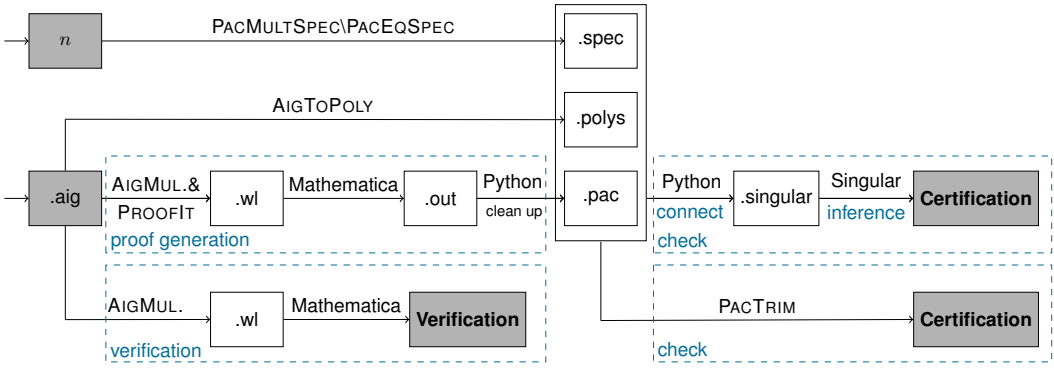
Proof Checking Algorithm

input G sequence of given polynomials
$r_1 \cdots r_k$ sequence of PAC proof rules

output "incorrect", "correct-proof", or "correct-refutation"

$P_0 \leftarrow G$
for $i \leftarrow 1 \ldots k$
  let $r_i = (o_i, v_i, w_i, p_i)$
  case $o_i = +$
    if $v_i \in P_{i-1} \ \wedge \ w_i \in P_{i-1} \ \wedge \ p_i = v_i + w_i$ **then** $P_i \leftarrow \text{append}(P_{i-1}, p_i)$
    else return "incorrect"
  case $o_i = *$
    if $v_i \in P_{i-1} \ \wedge \ p_i = v_i * w_i$ **then** $P_i \leftarrow \text{append}(P_{i-1}, p_i)$
    else return "incorrect"
for $i \leftarrow 1 \ldots k$
  if p_i is a non zero constant polynomial **then return** "correct-refutation"
return "correct-proof"

Toolflow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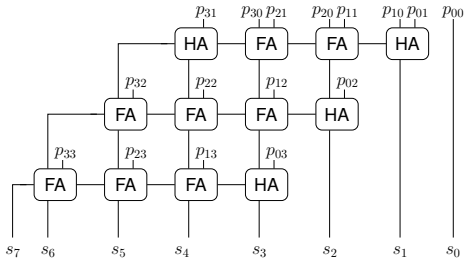
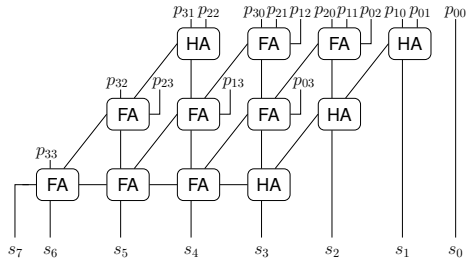
Correctness and Equivalence Checking





Tailored heuristics:

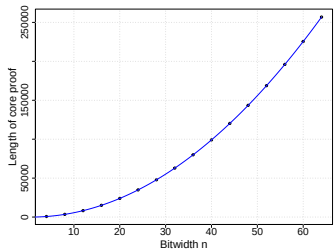
- Split verification problem into sub-problems $\rightarrow$ incremental verification [FMCAD'17]
- Rewrite and simplify Gröbner basis $\rightarrow$ adder-rewriting [DATE'18]

Verifying Correctness of FMCAD'17 and DATE'18 Results

n	mult	option	verification	proof generation	PACTRIM	Python/ Singular	length	core
16	btor	inc	4	70	0	4	14998	64%
16	btor	inc-add	1	37	0	4	12738	61%
16	btor	noninc	4	78	0	10	14966	64%
32	btor	inc	44	1631	1	83	63254	64%
32	btor	inc-add	7	801	1	67	53122	61%
32	btor	noninc	65	1811	5	551	63190	64%
64	btor	inc	622	49638	4	5125	259606	63%
64	btor	inc-add	121	22378	4	4650	216834	61%
64	btor	noninc	MO	MO	-	-	-	-
16	sparrc	inc	8	134	0	7	17445	62%
16	sparrc	inc-add	1	112	0	20	17804	63%
16	sparrc	noninc	11	2696	0	17	17413	62%
32	sparrc	inc	104	3582	1	175	73301	62%
32	**sparrc**	**inc-add**	**8**	**2611**	**2**	**457**	**75244**	**63%**
32	sparrc	noninc	351	TO	-	-	-	-
64	sparrc	inc	1575	TO	-	-	-	-
64	sparrc	inc-add	133	80906	12	EE	309164	62%
64	sparrc	noninc	MO	-	-	-	-	-

Equivalence Checking - DATE'18

n	mult	verification	proof generation	PACTRIM	Python/ Singular	length	core
16	btor-btor	2	75	1	14	25410	59%
32	btor-btor	27	1632	3	277	106114	59%
64	btor-btor	502	45155	15	EE	433410	59%
16	btor-sparrc	3	148	1	48	30476	61%
32	btor-sparrc	28	3456	7	1011	128236	60%
16	sparrc-sparrc	2	223	2	82	35542	61%
32	sparrc-sparrc	29	5363	11	1899	150358	61%



Proof length: $\mathcal{O}(n^2)$

[BeameLiew-CAV'17]: $\mathcal{O}(n^7 \log n)$
upper bound for resolution proof size

Conclusion & Future Work

Conclusion:

- Practical proof checking for algebraic reasoning
- Translation of polynomial calculus to a more concrete proof format
- Certification of correctness of certain multipliers

Future Work:

- Speed-up proof generation
- Explore connection between PAC and clausal proof systems (RUP, DRAT)

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Engineering

Use `PolynomialReduce` for reduction $f \xrightarrow{G} 0$ and deriving co-factors $h_1, ..., h_k$ such that $f = h_1 g_1 + \ldots + h_k g_k$.

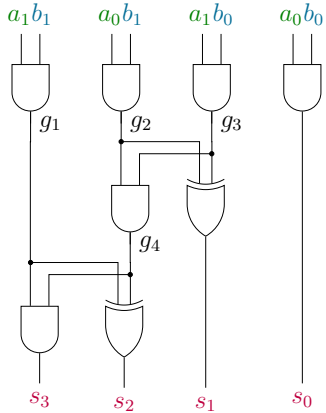
First we generate a multiplication proof rule for each product $h_i g_i$.

$$* : g_1, h_1, h_1 g_1; \qquad \cdots \qquad * : g_k, h_k, h_k g_k;$$

These products are now simply added together as follows:

$$\begin{array}{lll} + : & h_1 g_1, & h_2 g_2, \quad h_1 g_1 + h_2 g_2; \\ + : & h_1 g_1 + h_2 g_2, & h_3 g_3, \quad h_1 g_1 + h_2 g_2 + h_3 g_3; \\ & & \vdots \\ + : & h_1 g_1 + \ldots + h_{k-1} g_{k-1}, & h_k g_k, \quad f; \end{array}$$

Circuit Verification



$$
\begin{aligned}
G = \{ \quad & -s_3 + g_1 g_4, \\
& -s_2 + g_1 + g_4 - 2g_1 g_4, \\
& -g_4 + g_2 g_3, \\
& -s_1 + g_2 + g_3 - 2g_2 g_3, \\
& -g_1 + a_1 b_1, \\
& -g_2 + a_0 b_1, \\
& -g_3 + a_1 b_0, \\
& -s_0 + a_0 b_0, \\
& -a_1^2 + a_1, -a_0^2 + a_0, \\
& -b_1^2 + b_1, -b_0^2 + b_0 \}
\end{aligned}
$$

For all possible $a_i, b_i \in \mathbb{B}$: $(2a_1 + a_0) * (2b_1 + b_0) = 8s_3 + 4s_2 + 2s_1 + s_0$?